MATH 221 Lecture 36, December 6, 2000 ①

Limits

When a limit looks like it is coming out to $\frac{0}{0}$ it could really be coming out to anything.

Example $\lim_{x \to 0} \frac{5x}{x} = \frac{0}{0}$?? BAD

$$\lim_{x \to 0} \frac{5x}{x} = \lim_{x \to 0} 5 = 5.$$

Example $\lim_{x \to 0} \frac{642x}{x} = \frac{0}{0}$??? BAD

$$\lim_{x \to 0} \frac{642x}{x} = \lim_{x \to 0} 642 = 642.$$

Example $\lim_{x \to 0} \frac{x^2}{x} = \frac{0}{0}$?? BAD

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0.$$

Example $\lim_{x \to 0} \frac{x}{x^2} = \frac{0}{0}$??? BAD

$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x} =$ UNDEFINED since

$$\lim_{x \to 0^+} \frac{x}{x^2} = \lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{x}{x^2} = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Bad limits $\frac{0}{0}$ and $\frac{\infty}{\infty}$ and L'Hopital's rule ②

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$??? BAD

or $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$??? BAD

then sometimes it works to use

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example $\lim_{x \to 1} \frac{\ln x}{x-1}$

First: $\lim_{x \to 1} \frac{\ln x}{x-1} = \frac{0}{0}$??? BAD

Try L'Hopital's rule:

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1.$$

Example $\lim_{x \to \infty} \frac{\ln x}{e^x}$

First: $\lim_{x \to \infty} \frac{\ln x}{e^x} = \frac{\infty}{\infty}$??? BAD

Try L'Hopital's rule:

$$\lim_{x \to \infty} \frac{\ln x}{e^x} = \lim_{x \to \infty} \frac{1/x}{e^x} = \lim_{x \to \infty} \frac{1}{xe^x} = 0.$$

Example $\lim_{x\to\infty} \frac{(\ln x)^3}{x^2}$

First: $\lim_{x\to\infty} \frac{(\ln x)^3}{x^2} = \frac{\infty}{\infty}$??? BAD

Try L'Hopital's rule:

$$\lim_{x\to\infty} \frac{(\ln x)^3}{x^2} = \lim_{x\to\infty} \frac{3(\ln x)^2 \frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{3(\ln x)^2}{2x^2} = \frac{\infty}{\infty} ???$$

Try again:

$$\lim_{x\to\infty} \frac{3(\ln x)^2}{2x^2} = \lim_{x\to\infty} \frac{6(\ln x)\frac{1}{x}}{4x} = \lim_{x\to\infty} \frac{6\ln x}{4x^2} = \frac{\infty}{\infty} ???$$

Try again:

$$\lim_{x\to\infty} \frac{6\ln x}{4x^2} = \lim_{x\to\infty} \frac{6\frac{1}{x}}{8x} = \lim_{x\to\infty} \frac{6}{8x^2} = 0.$$

Why does L'Hopital's rule work?

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$??? then $f(a)=0$ and $g(a)=0$.

Let $x = a+\Delta x$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)}{g(a+\Delta x)} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{g(a+\Delta x)-g(a)}$$

$$= \lim_{\Delta x\to 0} \frac{\frac{f(a+\Delta x)-f(a)}{\Delta x}}{\frac{g(a+\Delta x)-g(a)}{\Delta x}} = \frac{\left.\frac{df}{dx}\right|_{x=a}}{\left.\frac{dg}{dx}\right|_{x=a}} = \lim_{x\to a} \frac{df/dx}{dg/dx}.$$

So $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\text{rate } f(x) \text{ approaches } 0}{\text{rate } g(x) \text{ approaches } 0}$

$$= \lim_{x\to a} \frac{df/dx}{dg/dx}.$$

BAD LIMITS $\frac{0}{0}$ and $\frac{\infty}{\infty}$

Example $\lim_{x\to 0} \frac{5x}{3x} = \frac{0}{0}$??? BAD

$\lim_{x\to 0} \frac{5x}{3x} = \lim_{x\to 0} 5/3 = 5/3$, since

$\frac{\text{rate } 5x \text{ goes to } 0}{\text{rate } 3x \text{ goes to } 0} = \frac{\frac{d\,5x}{dx}}{\frac{d\,3x}{dx}} = 5/3$, i.e. $5x$ goes to 0 $5/3$ as fast as $3x$ goes to 0.

Example $\lim_{x\to\infty} \frac{5x}{x} = \frac{\infty}{\infty}$??? BAD.

$\lim_{x\to\infty} \frac{5x}{x}$ Here $5x$ goes to ∞ 5 times as fast as x goes to ∞.

$$\lim_{x\to\infty} \frac{5x}{x} = \lim_{x\to\infty} 5 = 5.$$

Example $\lim_{x\to 0} \frac{\frac{1}{x^2}}{\frac{1}{x^3}} = \frac{\infty}{\infty}$???

$$\lim_{x\to 0} \frac{\frac{1}{x^2}}{\frac{1}{x^3}} = \lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0.$$

Here $\lim_{x\to 0} \frac{x^3}{x^2} = \frac{0}{0}$??? but x^3 goes to 0 much faster than x^2 goes to 0.

So $\lim_{x\to 0} \frac{x^3}{x^2} = 0$.

Example $\lim_{x\to 0} \frac{-\ln x}{1/x} = \frac{\infty}{\infty}$???

$$\lim_{x\to 0} \frac{-\ln x}{1/x} = \lim_{x\to 0} \frac{\text{rate } -\ln x \text{ goes to } \infty}{\text{rate } 1/x \text{ goes to } \infty}$$

$$= \lim_{x\to 0} \frac{-1/x}{-1/x^2} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

Other bad limits $\infty - \infty$, $0\cdot\infty$, 1^∞, 0^0

(a) $0\cdot\infty$

Example $\lim_{x\to\pi} (x-\pi)\cot x = 0\cdot\infty$???

$$\lim_{x\to\pi} (x-\pi)\cot x = \lim_{x\to\pi} \frac{x-\pi}{\tan x} = \frac{0}{0} \text{ ???}$$

$$\lim_{x\to\pi} \frac{x-\pi}{\tan x} = \lim_{x\to\pi} \frac{\text{rate that } x-\pi \text{ goes to } 0}{\text{rate that } \tan x \text{ goes to } 0}$$

$$= \lim_{x\to\pi} \frac{1}{\sec^2 x} = \lim_{x\to\pi} \cos^2 x = \cos^2\pi = (-1)^2 = 1.$$

(b) 1^∞

Example: $\lim_{x\to 0} (1-2x)^{1/x} \overset{?}{=} 1^\infty$??? BAD

$$\lim_{x\to 0} (1-2x)^{1/x} = \lim_{x\to 0} \left(e^{\ln(1-2x)}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1-2x)}$$

$$= \lim_{x\to 0} e^{-2\frac{\ln(1-2x)}{-2x}} = e^{-2\cdot 1} = e^{-2}.$$

(c) 0^0

Example $\lim_{x\to 0} x^x = 0^0$??? BAD

$$\lim_{x\to 0} x^x = \lim_{x\to 0} \left(e^{\ln x}\right)^x = \lim_{x\to 0} e^{x\ln x} = \lim_{x\to 0} e^{\frac{\ln x}{1/x}}$$

$$= e^{\frac{\infty}{\infty}} \text{ ??? BAD}$$

$$\lim_{x\to 0} e^{\frac{\ln x}{1/x}} = \lim_{x\to 0} e^{\frac{1/x}{-1/x^2}} = \lim_{x\to 0} e^{-\frac{x^2}{x}} = \lim_{x\to 0} e^{-x} = e^{-0} = \underline{1}.$$ (7)

(d) $\infty - \infty$

<u>Example</u> $\lim_{x\to 0} \frac{1}{x} - \frac{1}{x} = \infty - \infty$??? <u>BAD</u>

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x} = \lim_{x\to 0} 0 = 0.$$

<u>Example</u> $\lim_{x\to 0} x^{-1} - \csc x = \infty - \infty$??? <u>BAD</u>

$$\lim_{x\to 0} x^{-1} - \csc x = \lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \frac{0}{0} ???$$

Try L'Hopital:

$$\lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} = \frac{0}{0} ???$$

Try again:

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{-0}{1+1-0} = \frac{-0}{2} = 0.$$

<u>An example of when L'Hopitals rule doesn't work</u> (8)

<u>Example</u> $\lim_{x\to 0} \frac{x}{-(\ln x)^{-1}} = \frac{0}{0}$???

$$\lim_{x\to 0} \frac{x}{-(\ln x)^{-1}} = \lim_{x\to 0} \frac{1}{+(\ln x)^{-2}\frac{1}{x}} = \lim_{x\to 0} \frac{x}{(\ln x)^{-2}} = \frac{0}{0} ??$$

Try again:

$$\lim_{x\to 0} \frac{x}{(\ln x)^{-2}} = \lim_{x\to 0} \frac{1}{-2(\ln x)^{-3}\frac{1}{x}} = \lim_{x\to 0} \frac{x}{-2(\ln x)^{-3}} = \frac{0}{0} ???$$

Try again:

$$\lim_{x\to 0} \frac{x}{-2(\ln x)^{-3}} = \lim_{x\to 0} \frac{1}{6(\ln x)^{-4}\frac{1}{x}} = \lim_{x\to 0} \frac{x}{6(\ln x)^{-4}} = \frac{0}{0} ???$$

$\vdots$

Goes on forever ☹

Instead do

$$\lim_{x\to 0} \frac{x}{-(\ln x)^{-1}} = \lim_{x\to 0} -x\ln x = \lim_{x\to 0} \frac{-\ln x}{1/x} = \frac{\infty}{\infty} ???$$

Then

$$\lim_{x\to 0} \frac{-\ln x}{1/x} = \lim_{x\to 0} \frac{-1/x}{-1/x^2} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$