MATH 221, Lecture 31 November 22, 2000

Averages

Average of a bunch of numbers:

(a) Add up the numbers

(b) Divide by the number of numbers.

Example The average of $1, 2, 3, \ldots, 100$

$$\begin{array}{c} 1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100 \\ 100 + 99 + 98 + 97 + \cdots + 4 + 3 + 2 + 1 \\ \hline 101 + 101 + 101 + 101 + \cdots + 101 + 101 + 101 + 101 = 10100 \end{array}$$

So $1 + 2 + 3 + \cdots + 100 = \dfrac{10100}{2} = 5050$

So the average is $\dfrac{1 + 2 + \cdots + 100}{100} = \dfrac{5050}{100} = 50.5$.

Example Compute the average of

$$1, \tfrac{1}{3}, \tfrac{1}{3^2}, \tfrac{1}{3^3}, \ldots, \tfrac{1}{3^{50}}$$

$$\begin{aligned} (1 + x + x^2 + \cdots + x^{50})(1 - x) &= 1 + x + x^2 + x^3 + \cdots + x^{50} \\ &\quad - x - x^2 - x^3 - \cdots - x^{50} - x^{51} \\ &= 1 - x^{51} \end{aligned}$$

So $1 + x + x^2 + \cdots + x^{50} = \dfrac{1 - x^{51}}{1 - x}$

So $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \left(\frac{1}{3}\right)^{50} = \dfrac{1 - \left(\frac{1}{3}\right)^{51}}{1 - \frac{1}{3}} = \dfrac{1 - \frac{1}{3^{51}}}{\frac{2}{3}} = \dfrac{3 - \frac{1}{3^{50}}}{2}$

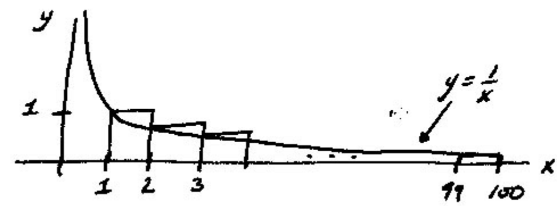
So $\dfrac{1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{50}}}{51} = \dfrac{\frac{3 - \frac{1}{3^{50}}}{2}}{51} = \dfrac{3 - \frac{1}{3^{50}}}{102} \approx \dfrac{3}{102} \approx .03$

Estimating the average

What is the average of $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$?

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99}$ is the area of the boxes

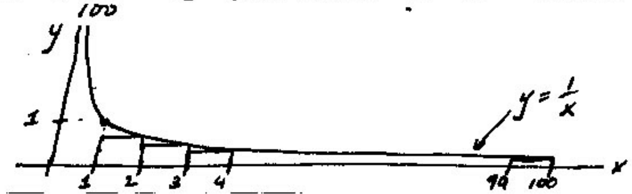
in the picture



So $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} \geq \displaystyle\int_1^{100} \frac{1}{x}\,dx = \ln x \Big|_{x=1}^{x=100} = \ln 100$

Next, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$ is the area of the boxes

in the picture

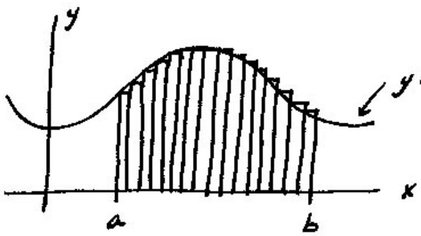


So $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100} \le \int_1^{100} \frac{1}{x}\,dx = \ln x \Big|_1^{100} = \ln 100.$

So $\dfrac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}}{100} \ge \dfrac{\ln 100+\frac{1}{100}}{100} = \dfrac{\ln 100}{100}+\dfrac{1}{100^2}$

and $\dfrac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}}{100} \le \dfrac{\ln 100+1}{100} = \dfrac{\ln 100}{100}+\dfrac{1}{100}$

Average value of a function



Average value of f from a to b = Average height of little boxes

$$= \lim_{\Delta x \to 0} \frac{f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x}{(\text{number of boxes})\,\Delta x}$$

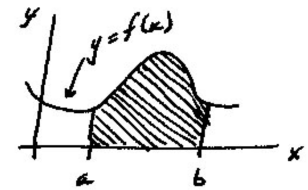
$$= \lim_{\Delta x \to 0} \frac{f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x}{b-a}$$

$$= \frac{\int_a^b f(x)\,dx}{b-a} = \frac{(\text{Area under } f \text{ from } a \text{ to } b)}{b-a}$$

So

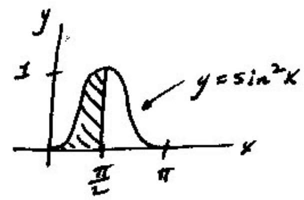
$$(\text{Average value})(b-a) = \frac{\text{Area under } f(x) \text{ from } a \text{ to } b}{b-a}.$$

has the same area as



A = average value of f from a to b

Example Find the average value of $f(x) = \sin^2 x,\ 0 \le x \le \pi/2$.



Average value is

$$\frac{\int_0^{\frac{\pi}{2}} f(x)\,dx}{\frac{\pi}{2}-0} = \frac{\int_0^{\frac{\pi}{2}} \sin^2 x\,dx}{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin^2 x + \sin^2 x)\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos^2 x+\sin^2 x)\,dx$$

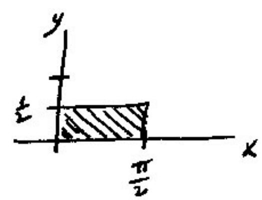
(5)

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1-(\cos^2 x - \sin^2 x))\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (1-\cos 2x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\frac{\pi}{2}} = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\sin 2\frac{\pi}{2}}{2}\right) - \frac{1}{\pi}\left(0 - \frac{\sin 0}{2}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}.$$

So 

is the same area as