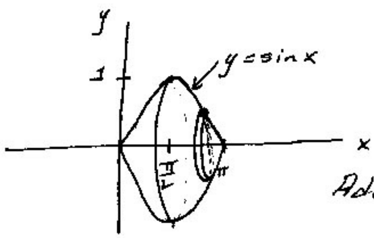
MATH 221 Lecture 29, November 15, 2000 ①

Example Find the volume generated when the area bounded by $y = \sin x$, $0 \le x \le \pi$, and $y = 0$ is rotated about the x-axis.



Slice: 

Volume of slice: $\pi R^2 dx$

Add up slices from $x=0$ to $x=\pi$.

$$\int_{x=0}^{x=\pi} \pi R^2 dx = \int_{x=0}^{x=\pi} \pi y^2 dx = \int_{x=0}^{x=\pi} \pi \sin^2 x \, dx$$

$$= \int_{x=0}^{x=\pi} \frac{\pi}{2}(\sin^2 x + \sin^2 x)\,dx = \int_{x=0}^{x=\pi} \frac{\pi}{2}(\sin^2 x + 1 - \cos^2 x)\,dx$$

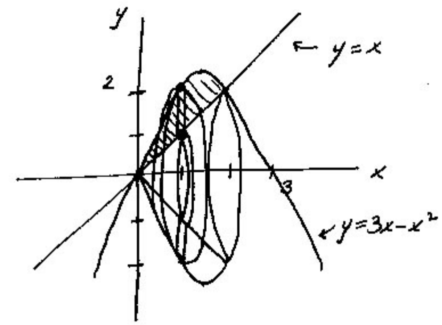
$$= \int_{x=0}^{x=\pi} \frac{\pi}{2}(1 - (\cos^2 x - \sin^2 x))\,dx = \int_{x=0}^{x=\pi} \frac{\pi}{2}(1 - \cos 2x)\,dx$$

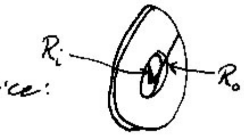
$$= \frac{\pi}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_{x=0}^{x=\pi} = \frac{\pi}{2}\left(\pi - \frac{\sin 2\pi}{2}\right) - \frac{\pi}{2}\left(0 - \frac{\sin 0}{2}\right)$$

$$= \frac{\pi}{2}(\pi - 0) = \frac{\pi^2}{2}$$

②

Example Find the volume generated by rotating the area bounded by the curves $y = 3x - x^2$ and $y = x$ about the x-axis.



Slice: 

Volume of slice: $(\pi R_o^2 - \pi R_i^2)\,dx$

Add slices from $x=0$ to $x=2$.

$$\int_{x=0}^{x=2} (\pi R_o^2 - \pi R_i^2)\,dx = \int_{x=0}^{x=2} (\pi y_{top}^2 - \pi y_{bottom}^2)\,dx$$

$$= \int_{x=0}^{x=2} (\pi(3x-x^2)^2 - \pi x^2)\,dx = \int_{x=0}^{x=2} \pi(9x^2 - 6x^3 + x^4 - x^2)\,dx$$

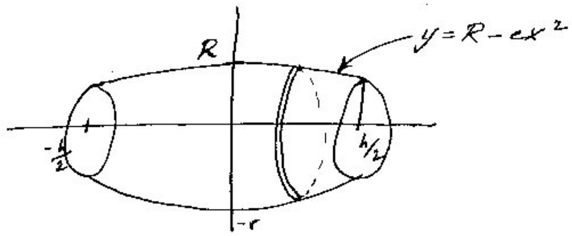
$$= \int_{x=0}^{x=2} \pi(8x^2 - 6x^3 + x^4)\,dx = \pi\left(\frac{8x^3}{3} - \frac{6x^4}{4} + \frac{x^5}{5}\right)\Big|_{x=0}^{x=2}$$

$$= \pi\left(\frac{8\cdot 8}{3} - \frac{6\cdot 2^4}{4} + \frac{2^5}{5}\right) - \pi(0 - 0 + 0) = \pi 2^5\left(\frac{2}{3} - \frac{3}{4} + \frac{1}{5}\right)$$

$$= 32\pi\left(\frac{40}{60} - \frac{45}{60} + \frac{12}{60}\right) = \frac{32\pi\cdot 7}{60} = \frac{8\cdot 7\pi}{15} = \frac{56\pi}{15}$$

$②$

Example A barrel of height h and maximum radius R is constructed by rotation of the parabola $y = R - cx^2$, $-h/2 \le x \le h/2$.



Volume of a slice

is $\pi y^2 dx$

Add up slices from $x = -h/2$ to $h/2$

$$\int_{x=-h/2}^{x=h/2} \pi y^2 dx = \int_{x=-h/2}^{x=h/2} \pi (R - cx^2)^2 dx$$

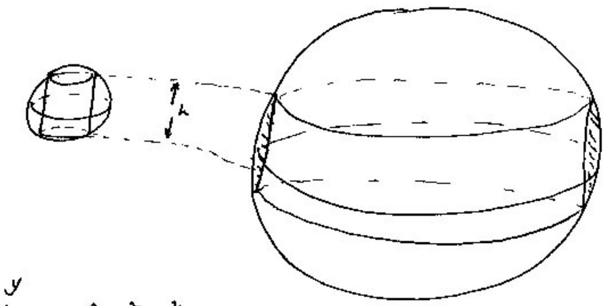
$$= \int_{x=-h/2}^{x=h/2} \pi (R^2 - 2cRx^2 + c^2x^4)\, dx = \pi\left(R^2x - 2cR\frac{x^3}{3} + c^2\frac{x^5}{5}\right)\Bigg|_{x=-h/2}^{x=h/2}$$

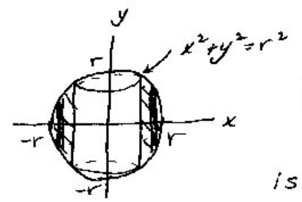
$$= \pi\left(R^2\frac{h}{2} - \frac{2cR}{3}\frac{h^3}{8} + \frac{c^2}{5}\frac{h^5}{2^5}\right) - \pi\left(R^2\frac{(-h)}{2} + \frac{2cR}{3}\frac{h^3}{2^3} + \frac{(-c^2h^5)}{5\cdot 2^5}\right)$$

$$= \pi\left(R^2h - \frac{4cRh^3}{3\cdot 8} + \frac{2c^2h^5}{5\cdot 2^5}\right) = \pi\left(R^2h - \frac{cRh^3}{6} + \frac{c^2h^5}{5\cdot 16}\right).$$

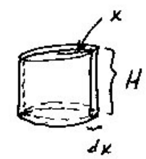
$③$

Example You are given two spherical balls of wood, one of radius r and a second one of radius R. A circular hole is bored through each ball and the resulting napkin rings have height h. Which napkin ring contains more wood?





Volume of a slice



is $2\pi x H dx$.

Add slices from $x = \sqrt{r^2 - (\frac{1}{2}h)^2}$ to $x = r$.

$$\int_{x=\sqrt{r^2-\frac{h^2}{4}}}^{x=r} 2\pi x\, 2y\, dx = \int_{x=\sqrt{r^2-\frac{h^2}{4}}}^{x=r} 4\pi x \sqrt{r^2-x^2}\, dx$$

$$= \int_{x=\sqrt{r^2-\frac{h^2}{4}}}^{x=r} (-2\pi)(-2x)(r^2-x^2)^{\frac{1}{2}} dx = -2\pi\, (r^2-x^2)^{3/2}\,\frac{2}{3}\Bigg|_{x=\sqrt{r^2-\frac{h^2}{4}}}^{x=r}$$

(4)

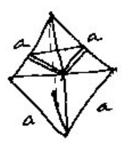
$$= -2\pi(r^2-r^2)^{3/2}\frac{2}{3} - (-2\pi)\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{3/2}\frac{2}{3}$$

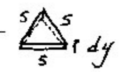
$$= 0 + 2\pi\left(\frac{h^2}{4}\right)^{3/2}\frac{2}{3} = 2\pi\left(\frac{h}{2}\right)^3\frac{2}{3} = \frac{4\pi}{3}\frac{h^3}{8} = \frac{\pi h^3}{8}.$$

This doesn't depend on r!!

So both napkin rings contain the same amount of wood.

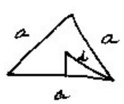
Example Find the volume of a tetrahedron where each side of the tetrahedron is an equilateral triangle with side length a.



Volume of a slice 

is $dy\frac{(\text{base})(\text{height})}{2} = \frac{(\frac{1}{2}s)(\frac{\sqrt{3}}{2}s)}{2}dy$

Add up slices from $y=0$ to $y=$ (height of tetrahedron $= H$



$\frac{\frac{1}{2}a}{d} = \cos 30° = \frac{\sqrt{3}}{2}$

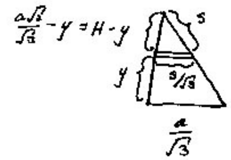
So $d = \frac{\frac{1}{2}a}{\frac{\sqrt{3}}{2}} = \frac{a}{\sqrt{3}}$. So $H = \sqrt{a^2-\left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{a^2-\frac{a^2}{3}}$

$$= \sqrt{\frac{2a^2}{3}} = \frac{a\sqrt{2}}{\sqrt{3}}.$$

(5)

So we want

$$2\int_{y=0}^{y=H}\frac{(\frac{1}{2}s)(\frac{\sqrt{3}}{2}s)}{2}\,dy = \int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}\,s^2}{8}\,dy.$$



But $H-y = \frac{\sqrt{2}}{\sqrt{3}}s$.

So $y = H - \frac{\sqrt{2}}{\sqrt{3}}s$.

So $\frac{dy}{ds} = -\frac{\sqrt{2}}{\sqrt{3}}$.

$$\int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}}{8}s^2\,dy = \int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}}{8}s^2\frac{dy}{ds}\,ds = \int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}}{8}s^2\left(-\frac{\sqrt{2}}{\sqrt{3}}\right)ds$$

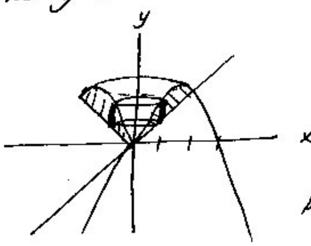
$$= 2\cdot\frac{\sqrt{3}}{8}\,\frac{s^3}{3}\left(-\frac{\sqrt{2}}{\sqrt{3}}\right)\Bigg|_{y=0}^{y=H} = \frac{-\sqrt{2}\cdot 2}{8\cdot 3}\,s^3\Bigg|_{s=a}^{s=0}$$

$$= \left(-\frac{\sqrt{2}\cdot 2}{8\cdot 3}0^3\right) - \left(-\frac{\sqrt{2}\cdot 2}{8\cdot 3}a^3\right) = \frac{\sqrt{2}\cdot 2}{24}a^3 = \frac{\sqrt{2}}{12}a^3$$

Since $H = \frac{a\sqrt{2}}{\sqrt{3}}$ we can also write this as

$$\frac{\sqrt{2}}{12}a^3 = \left(\frac{\sqrt{2}}{\sqrt{3}}a\right)\frac{\sqrt{3}}{12}a^2 = H\frac{\sqrt{3}\,a^2}{12} = \frac{\sqrt{3}\,a^2 H}{12}.$$

<u>Example</u> Find the volume generated by rotating the area bounded by $y = 3x - x^2$ and $y = x$ about the <u>y-axis</u>.



Volume of a slice

is $2\pi R H dx$

Add slices from $x=0$ to $x=2$.

$$\int_{x=0}^{x=2} 2\pi R H\,dx = \int_{x=0}^{x=2} 2\pi x\,(y_{upper} - y_{lower})\,dx$$

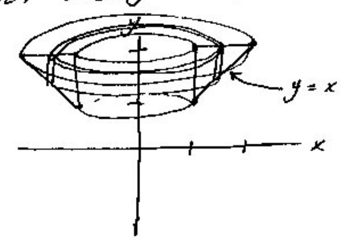
$$= \int_{x=0}^{x=2} 2\pi x(3x - x^2 - x)\,dx = \int_{x=0}^{x=2} 2\pi x(2x - x^2)\,dx$$

$$= \int_{x=0}^{x=2} 2\pi(2x^2 - x^3)\,dx = 2\pi\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_{x=0}^{x=2}$$

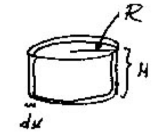
$$= 2\pi\left(\frac{2}{3}\cdot 8 - \frac{16}{4}\right) - 2\pi(0-0) = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$$

$$= 2\pi\cdot 16\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{16\pi}{6} = \frac{8\pi}{3}.$$

<u>Example</u> Find the volume generated by revolving the triangle with vertices $(1,1)$, $(1,2)$ and $(2,2)$ about the y-axis.



Volume of a slice



is $2\pi R H dx$

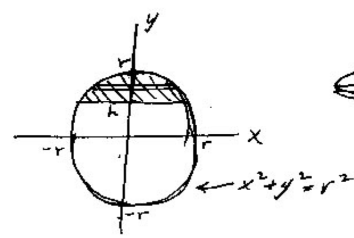
Add slices from $x=1$ to $x=2$.

$$\int_{x=1}^{x=2} 2\pi R H\,dx = \int_{x=1}^{x=2} 2\pi x(2-x)\,dx = \int_{1}^{2} 2\pi(2x - x^2)\,dx$$

$$= 2\pi\left(x^2 - \frac{x^3}{3}\right)\Bigg|_{x=1}^{x=2} = 2\pi\left(4 - \frac{8}{3}\right) - 2\pi\left(1 - \frac{1}{3}\right)$$

$$= 2\pi\left(\frac{4}{3}\right) - 2\pi\left(\frac{2}{3}\right) = \frac{4\pi}{3}.$$

<u>Example</u> Find the volume of a slice obtained by chopping off the end of a sphere of radius r, if the slice has thickness h (at its thickest point).



Volume of a slice 

is $\pi R^2 dy$

Add slices from $y=h$ to $y=\cancel{R}\,r$

$$\int_{y=h}^{y=r} \pi R^2\,dy = \int_{y=h}^{y=r} \pi x^2\,dy = \int_{y=h}^{y=r} \pi\,(r^2-y^2)\,dy$$

$$= \pi\left(r^2 y - \frac{y^3}{3}\right)\Bigg|_{y=h}^{y=r} = \pi\left(r^3 - \frac{r^3}{3}\right) - \pi\left(r^2 h - \frac{h^3}{3}\right)$$

$$= \pi\frac{2}{3}r^3 - \pi r^2 h + \frac{\pi h^3}{3} = \frac{\pi}{3}\left(2r^3 - 3r^2 h + h^3\right).$$