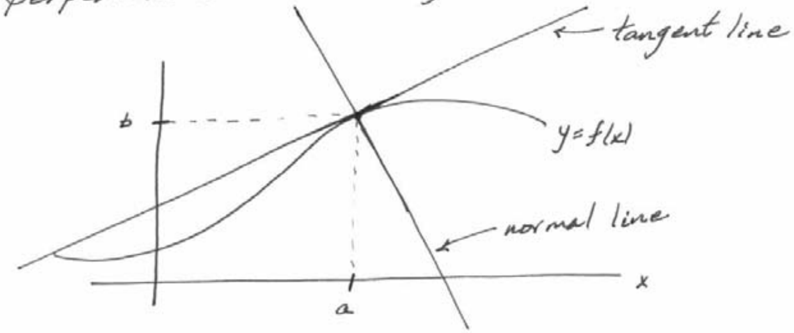
MATH 221 Lecture 19, October 23, 2000 ①

The *tangent line* to a curve $f(x)$ at the point (a,b) is the line through (a,b) with the same slope as $f(x)$ at the point (a,b).

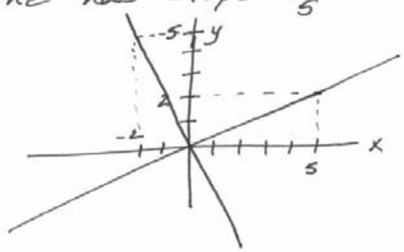
The *normal line* is the line through (a,b) which is perpendicular to the tangent line.



The slope of the tangent line at (a,b) is

$$\left.\frac{df}{dx}\right|_{x=a}.$$

If a line has slope $\frac{2}{5}$



then the perpendicular line has slope $\frac{5}{-2}$

Example Find the equations of the tangent and normal to the curve $y=x^4-6x^3+13x^2-10x+5$ at the point where $x=1$. ②

The slope of the tangent line at $x=1$ is

$$\left.\frac{df}{dx}\right|_{x=1} = \left.4x^3-18x^2+26x-10\right|_{x=1} = 4-18+26-10=2.$$

The tangent line goes through the point

$x=1$

$y=1-6+13-10+5=3.$

The equation of a line is $y=mx+b$ where m is the slope. So, for our line

$m=2$ and $3=m\cdot 1+b=2\cdot 1+b.$

So $b=1$. So the tangent line is

$y=2x+1.$

The slope of the normal line is $\frac{1}{-2}=-\frac{1}{2}$.

The equation of the normal line is $y=mx+b$ with $m=\frac{-1}{2}$ and $3=m\cdot 1+b=\frac{-1}{2}+b.$

So $b=\frac{7}{2}$ and $y=\frac{-1}{2}x+\frac{7}{2}$ is the normal line.

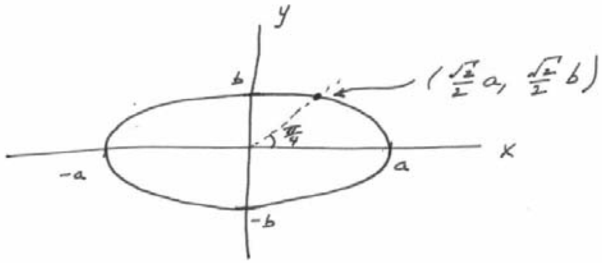
③

Example Find the equation of the tangent and normal lines to the curve

$$x = a\cos\theta, \quad y = b\sin\theta \quad \text{at } \theta = \frac{\pi}{4}.$$

First graph this:

$$\frac{x}{a} = \cos\theta, \quad \frac{y}{b} = \sin\theta. \quad \text{So} \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$



When $\theta = \frac{\pi}{4}$, $\quad x = a\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}a$

$$y = b\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}b$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{\substack{x=\frac{\sqrt{2}}{2}a \\ y=\frac{\sqrt{2}}{2}b}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{\frac{d\, b\sin\theta}{d\theta}}{\frac{d\, a\cos\theta}{d\theta}}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{b\cos\theta}{-a\sin\theta}\right|_{\theta=\frac{\pi}{4}}$$

$$= \frac{b\frac{\sqrt{2}}{2}}{-a\frac{\sqrt{2}}{2}} = \frac{-b}{a}.$$

④

So the equation of the tangent line is $y = mx + y_0$

with $m = \frac{-b}{a}$ and $\frac{\sqrt{2}}{2}b = m\frac{\sqrt{2}}{2}a + y_0 = \frac{-b}{a}\frac{\sqrt{2}}{2}a + y_0$.

So $y_0 = \frac{\sqrt{2}}{2}b + \frac{\sqrt{2}}{2}b = \sqrt{2}b$.

So the equation of the tangent line is

$$y = \frac{-b}{a}x + \sqrt{2}b.$$

The equation of the normal line is $y = mx + y_0$

with $m = \frac{a}{b}$ and $\frac{\sqrt{2}}{2}b = m\frac{\sqrt{2}}{2}a + y_0 = \frac{a}{b}\frac{\sqrt{2}}{2}a + y_0$

So $y_0 = \frac{\sqrt{2}}{2}b - \frac{\sqrt{2}}{2}\frac{a^2}{b} = \frac{\sqrt{2}}{2}\left(\frac{b^2 - a^2}{b}\right)$.

So the equation of the normal line is

$$y = \frac{a}{b}x + \frac{\sqrt{2}}{2}\left(\frac{b^2 - a^2}{b}\right)$$

Example Find the equations of the normal to $2x^2 - y^2 = 14$, parallel to the line $x + 3y = 4$.

The line $x + 3y = 4$ is the same as

$$y = \frac{-1}{3}x + \frac{4}{3}. \quad \text{So it has slope } \frac{-1}{3}.$$

So the slope of the normal line is $\frac{-1}{3}$.

So the slope of the tangent line is 3.

So $\left.\frac{dy}{dx}\right|_{x=a} = 3$.

Now $4x - 2y\frac{dy}{dx} = 0$. So $\frac{dy}{dx} = \frac{-4x}{-2y} = \frac{2x}{y}$.

So we want $\frac{2x}{y} = 3$ and $2x^2 - y^2 = 14$.

So $y = \frac{2}{3}x$ and $2x^2 - \left(\frac{2}{3}x\right)^2 = 14$.

So $2x^2 - \frac{4}{9}x^2 = 14$.

So $\frac{14}{9}x^2 = 14$. So $x^2 = 9$. So $x = \pm 3$.

So $x = 3$ and $y = \frac{2}{3}\cdot 3 = 2$ <u>or</u> $x = -3$ and $y = \frac{2}{3}(-3) = -2$.

In the first case:

The normal has slope $\frac{-1}{3}$ and goes through $(3,2)$.

So $m = \frac{-1}{3}$ and $2 = m\cdot 3 + y_0 = \frac{-1}{3}\cdot 3 + y_0$

So $y_0 = 3$ and the equation of the normal line is

$$y = \frac{-1}{3}x + 3.$$

In the second case:

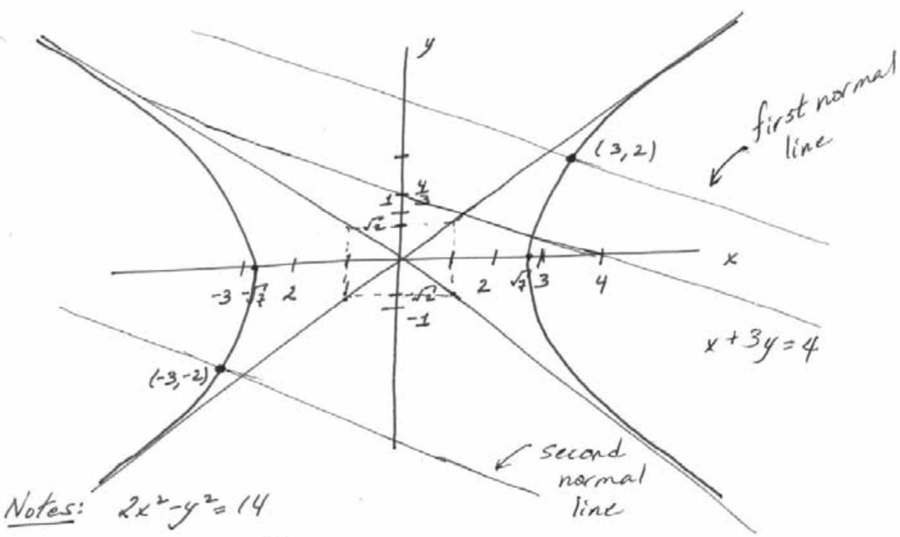
The normal has slope $\frac{-1}{3}$ and goes through $(-3,-2)$.

So $m = \frac{-1}{3}$ and $-2 = m(-3) + y_0 = \left(\frac{-1}{3}\right)(-3) + y_0$.

So $y_0 = -3$ and the equation of the normal line is

$$y = \frac{-1}{3}x - 3.$$

The graph should explain how there can be <u>two</u> normal lines parallel to $x + 3y = 4$



<u>Notes:</u> $2x^2 - y^2 = 14$

a) If $y = 0$, $x = \pm\sqrt{7}$

(b) $2 - \left(\frac{y}{x}\right)^2 = \frac{14}{x^2}$. So, as $x \to \infty$, this becomes $2 - \left(\frac{y}{x}\right)^2 = 0$.

$\left(\frac{y}{x}\right)^2 = 2$, $\left(\frac{y}{x}\right) = \pm\sqrt{2}$, $y = \pm\sqrt{2}\,x$.