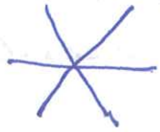
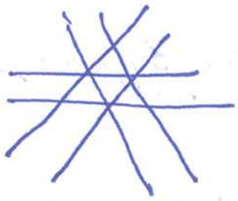
10.04.2025 (1)
Sydney SMRI
A. Ram

Hyperplanes

The braid arrangement $\mathcal{A}_0$ 

The Shi arrangement $\mathcal{A}_0 \cup \mathcal{A}_1$ 

The affine arrangement $\mathcal{A}_{\mathbb{Z}} = \bigcup_{\ell \in \mathbb{Z}} \mathcal{A}_\ell$.

Theorem (Type A)

(a) The number of chambers (conn. comps of $\mathbb{R}^n \setminus \mathcal{A}_0$) is $n!$

(b) The number of Shi regions (conn. comps of $\mathbb{R}^n \setminus (\mathcal{A}_0 \cup \mathcal{A}_1)$) is $D_{1+\frac{1}{n}} = (n+1)^{n-1}$.

(c) The number of dominant Shi regions is
$C_{1+\frac{1}{n}} = \frac{1}{n+1}\binom{2n}{n}$ (Catalan number).

More generally, for a crystallographic reflection group

$$|W_{fin}| = \prod_{i=1}^{r} d_i, \quad D_{1+\frac{1}{n}} = (h+1)^r, \quad C_{1+\frac{1}{n}} = \prod_{i=1}^{r} \frac{h+1+d_i}{d_i}$$

where $d_1, \ldots, d_r$ are the degrees

$h = d_r$ is the Coxeter number

r is the rank

Affine Weyl group

$\mathfrak{a}_1$ the fundamental alcove

$s_0, s_1, \ldots, s_r$ reflections in the walls of $\mathfrak{a}_1$

Then

$$w\mathfrak{a}_1 = \mathfrak{a}_w = \mathfrak{a}_{s_{i_1} \cdots s_{i_\ell}}, \text{ where}$$

$w = s_{i_1} \cdots s_{i_\ell}$ is a minimal length walk from $\mathfrak{a}_1$ to $\mathfrak{a}_w$.

Proposition $\bigcup_{\text{Shi regions}} \mathfrak{a}_{w^{-1}} = (h+1)$-dilate of $\mathfrak{a}_1$

Let S_w be a dominant Shi region, $y^\vee \in S_w$.

$$\mathfrak{m} = \mathfrak{a} \oplus \Big(\bigoplus_{\substack{\alpha \in R \\ \langle y^\vee, \alpha \rangle \geq -1}} \mathfrak{g}_\alpha \Big)$$ is a $\mathfrak{b}$-submodule of $\mathfrak{g}$ with $\mathfrak{m} \supseteq \mathfrak{b}$,

a Hessenberg space. The orthogonal complement of $\mathfrak{m}$ w.r.t. Killing form is $\mathfrak{m}^\perp$, an ad-nilpotent ideal of $\mathfrak{b}$

A Dyck path is $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2n}) \in \{1, -1\}^{2n}$ such that

(a) If $j \in \{1, \ldots, 2n\}$ then $\varepsilon_1 + \cdots + \varepsilon_j \leq 0$

(b) $\varepsilon_1 + \cdots + \varepsilon_{2n} = 0$.

DAHA modules

H double affine Hecke algebra (DAHA)

$\cup$

eHe spherical DAHA

depending on parameters q and t with

$$q^{h+1}t^h = 1.$$

Theorem

(a) H has a unique fin. dim'l simple module $L_{1+\frac{1}{h}}$, which has dimension $D_{1+\frac{1}{h}}$.

(b) eHe has a unique fin. dim'l simple module $eL_{1+\frac{1}{h}}$, which has dimension $C_{1+\frac{1}{h}}$.

Let

P_λ bosonic Macdonald polynomials

E_μ electronic Macdonald polynomials

Then

$eL_{1+\frac{1}{h}}$ has basis $\left\{ P_{w\cdot 0+\rho} \,\middle|\, \begin{array}{l} S_w \text{ is a dominant} \\ \text{Shi region} \end{array} \right\}$

$L_{1+\frac{1}{h}}$ has basis $\left\{ E_{w\cdot 0+\rho} \,\middle|\, \begin{array}{l} S_w \text{ is a} \\ \text{Shi region} \end{array} \right\}$

Affine Springer Fibers

$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F}((\epsilon))) = M^{n\text{-per}}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{F})$ n-periodic matrices $a_{ij} = a_{i+n,j+n}$.

$\cup|$

$\mathfrak{k} = \mathfrak{gl}_n(\mathbb{F}[[\epsilon]])$ block upper triangular

$\cup|$

$\mathfrak{b}$ upper triangular.

$G = GL_n(\mathbb{F}((\epsilon))) = \{g \in \mathfrak{g} \mid g^{-1} \in \mathfrak{g}\}$

$\cup|$

$K = GL_n(\mathbb{F}[[\epsilon]]) = \{g \in \mathfrak{k} \mid g^{-1} \in \mathfrak{k}\}$

$\cup|$

$I = \text{Iwahori} = \{g \in \mathfrak{b} \mid g^{-1} \in \mathfrak{b}\}$

$$G = \bigsqcup_{w \in W} IwI, \qquad W = \left\{\begin{matrix}n\text{-periodic}\\ \text{permutations}\end{matrix}\right\}$$

Let
$$\pi^{-1} = \sum_{i \in \mathbb{Z}} E_{i,i+1}$$

$$Y_{1+\frac{1}{n}} = \{gI \in G/I \mid g^{-1}\pi^{-(n+1)}g \in \mathfrak{b}\}$$

$$Z_{1+\frac{1}{n}} = \{g\cancel{K}K \in G/K \mid g^{-1}\pi^{-(n+1)}g \in \mathfrak{k}\}$$

Then
$$Y_{1+\frac{1}{n}} = \bigcup_{\text{Shi regions}} (Y_{1+\frac{1}{n}} \cap IwI)$$

$$Z_{1+\frac{1}{n}} = \bigcup_{\substack{\text{dominant}\\ \text{Shi regions}}} (Z_{1+\frac{1}{n}} \cap IwK)$$

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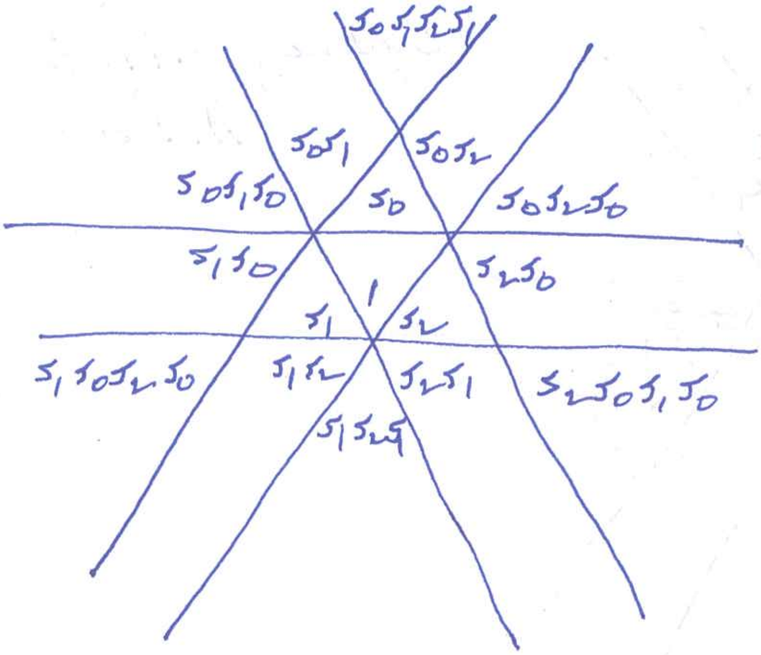

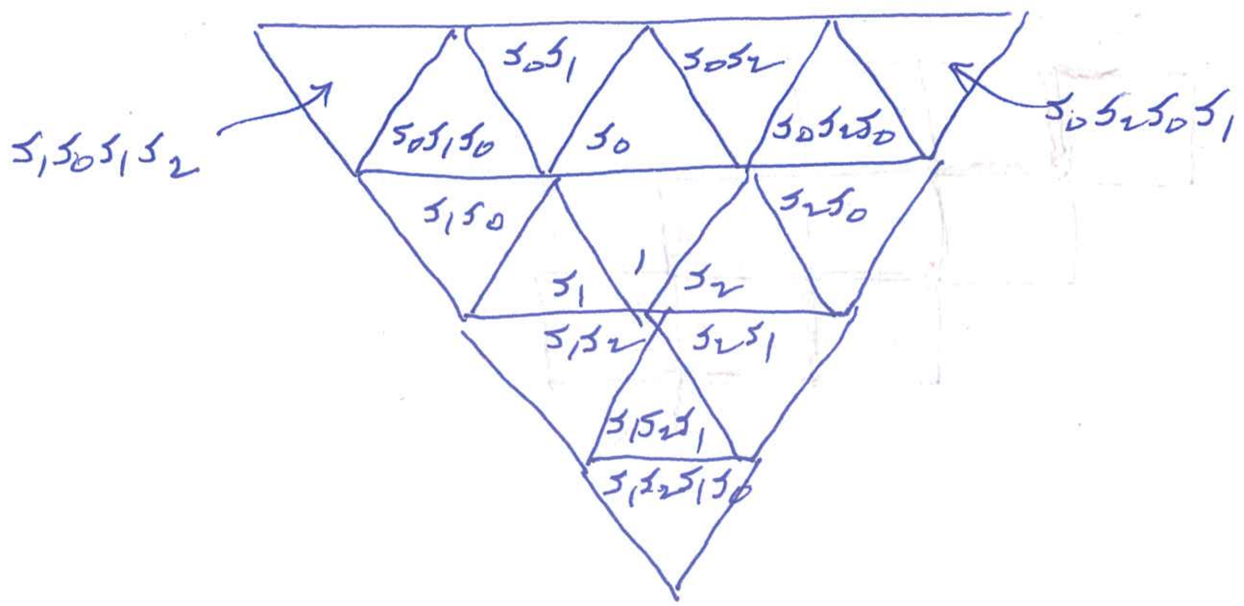

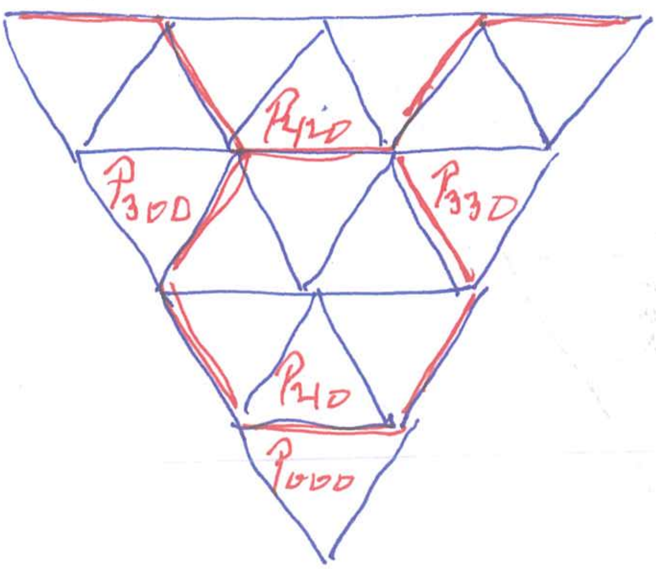

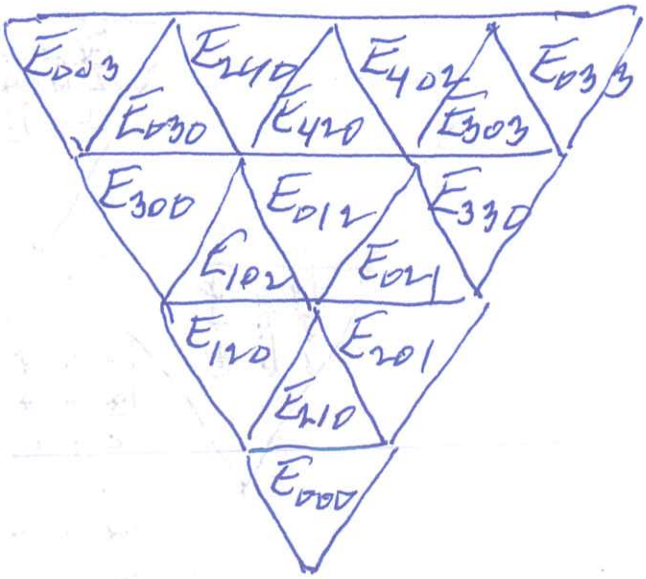

10.06.2025 (2a)
Sydney SMRI
A. Ram