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A. Ram

It's a numberful world

Part I: Clocks, freedom and addicts.

Clocks Email to students.

The first number system you learn as a newborn is the clock. Your parents are obsessed with it. They look at a round thing on the wall, say something about adding 1 or 2 and then make a life changing announcement about when you will eat or get a nappy change. They decide how long you will sleep from the clock and when they won't pay attention to you.

Very soon you learn that if you go down for a nap at 12 and sleep for 2 hours then you wake up at 2, just in time to change a nappy and go for a ride in the pram around the park.

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That's how you learn that

$$12+2=2.$$

Very soon you learn that there is a nappy change every 3 hours and that there are 4 nappy changes during the day and 4 nappy changes at night. That's how you learn that

$$3\cdot 4=12.$$

Professional mathematicians call 12 zero because $12+2=2$, and even though you know, since birth, that $3\cdot 4=0$, primary school teachers get all upset when you point out that two nonzero numbers can multiply together to give 0.

In the clock number system, what is the prime factorization of 6?

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P.S. In the French language the word for clock is montre.
In mathematics language the word for clock is $\mathbb{Z}/12\mathbb{Z}$ (zed mod 12 zed).

After a couple of tests with students, the evidence is that, in this case, the email is actually more effective for engaging students with this topic than a prerecorded video about the number system $\mathbb{Z}/12\mathbb{Z}$.

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Freedom

What is the definition of 3?

$$3 = 2+1 = 1+1+1.$$

$\mathbb{Z}_{>0}$ is the free monoid generated by 1.

$\mathbb{Z}_{\geq 0}$ is the free monoid with identity generated by 1.

$\mathbb{Z}$ is the free group generated by 1.

$\mathbb{Q}$ is the smallest field containing $\mathbb{Z}$.

Remarks

(1) Multiplication comes from

$\mathrm{Hom}(\mathbb{Z}_{>0}, \mathbb{Z}_{>0})$,

$\mathrm{Hom}(\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 0})$

$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$

$\left(\begin{array}{l} \varphi_a : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0} \\ \qquad x \longmapsto ax \\ \text{is a monoid morphism} \end{array}\right)$

(2) The order is defined by

$x \leq y$ if there exists $a \in \mathbb{Z}_{\geq 0}$ such that $x + a = y$.

(3) To prove a statement like

If $n \in \mathbb{Z}_{>0}$ then $\frac{dx^n}{dx} = nx^{n-1}$

one must access the definition of $\mathbb{Z}_{>0}$. The action of accessing the definition of $\mathbb{Z}_{>0}$ is called "proof by induction".

(4) A construction of $\mathbb{Q}$ is

$\mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}$ and $b \neq 0 \}$ with

$\frac{a}{b} = \frac{c}{d}$ if $ad = bc$ (this is an equivalence relation!!)

and the map
$$\begin{matrix} \mathbb{Z} & \to & \mathbb{Q} \\ n & \mapsto & \frac{n}{1} \end{matrix}.$$

(5) $SL_2(\mathbb{Z}_{\geq 0}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}_{\geq 0}) \,\middle|\, ad - bc = 1 \right\}$

with operation matrix multiplication.

Theorem $SL_2(\mathbb{Z}_{\geq 0})$ is the free monoid with identity generated by L and R.

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$SL_2(\mathbb{Z})$ is not a free group on two generators

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Addicts

The $\frac{1}{10}$-adics

$$\mathbb{Q}^{small}_{[0,10)} = \left\{ a_0 + a_1\tfrac{1}{10} + a_2\left(\tfrac{1}{10}\right)^2 + \cdots + a_\ell\left(\tfrac{1}{10}\right)^\ell \;\middle|\; \begin{array}{l} \ell\in\mathbb{Z}_{\geq 0} \\ a_i\in\{0,1,\ldots,9\}\end{array} \right\}$$
$$= \{ a_0.a_1a_2\ldots a_\ell \mid \ell\in\mathbb{Z}_{\geq 0},\ a_i\in\{0,1,\ldots,9\} \}$$

Addition is $\mathbb{Z}/10\mathbb{Z}$ with carries to the left

$$0.09 + 0.01 = \begin{array}{r} \overset{1}{0.09} \\ \underline{0.01} \\ 0.10 \end{array} = 0.10$$

$$\mathbb{R}_{[0,10)} = \left\{ a_0 + a_1\tfrac{1}{10} + a_2\left(\tfrac{1}{10}\right)^2 + \cdots \;\middle|\; a_i\in\{0,1,\ldots,9\} \right\}$$
$$= \{ a_0.a_1a_2\ldots \mid a_i\in\{0,1,\ldots,9\} \}$$

$$\mathbb{R} = \left\{ \pm\left(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots \right) \;\middle|\; \begin{array}{l} \ell\in\mathbb{Z} \text{ and} \\ a_i\in\{0,1,\ldots,9\}\end{array} \right\}$$
$$= \left\{ \pm\left(a_{-\ell}\cdots a_{-2}a_{-1}a_0.a_1a_2\cdots \right) \;\middle|\; \begin{array}{l} \ell\in\mathbb{Z} \text{ and} \\ a_i\in\{0,1,\ldots,9\}\end{array} \right\}$$

The 3-adics

$$\mathbb{Z}_{\geq 0} = \left\{ a_0 + a_1 3 + a_2 3^2 + \cdots + a_\ell 3^\ell \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{0,1,2\} \end{array} \right\}$$

$$= \left\{ a_0 . a_1 a_2 \ldots a_\ell \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ \text{and } a_i \in \{0,1,2\} \end{array} \right\}$$

Addition is $\mathbb{Z}/3\mathbb{Z}$ with carries to the right

$$0.2 + 0.1 = \begin{array}{r} 0.2\overset{1}{0} \\ \underline{0.10} \\ 0.01 \end{array} = 0.01$$

$$\mathbb{Z}_3 = \{ a_0 + a_1 3 + a_2 3^2 + \cdots \mid a_i \in \{0,1,2\} \}$$

$$= \{ a_0 . a_1 a_2 \ldots \mid a_i \in \{0,1,2\} \}$$

$$\mathbb{Q}_3 = \left\{ a_\ell 3^{-\ell} + a_{\ell+1} 3^{-\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \text{ and} \\ a_i \in \{0,1,2\} \end{array} \right\}$$

$$= \left\{ a_\ell \cdots a_2 a_1 a_0 . a_1 a_2 \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \text{ and} \\ a_i \in \{0,1,2\} \end{array} \right\}$$

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Polynomials

$$\mathbb{F}_3[t] = \left\{ a_0 + a_1 t + \cdots + a_\ell t^\ell \;\middle|\; \begin{matrix} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{0,1,2\} \end{matrix} \right\}$$

$$= \left\{ a_0 . a_1 a_2 \cdots a_\ell \;\middle|\; \begin{matrix} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{0,1,2\} \end{matrix} \right\}$$

Addition is $\mathbb{Z}/3\mathbb{Z}$ with no carries

$$0.2 + 0.1 = \begin{array}{r} 0.2 \\ 0.1 \\ \hline 0.0 \end{array} = 0.0$$

or $2t + t = 3t = 0t = 0$.

$$\mathbb{F}_3[[t]] = \{ a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \{0,1,2\} \}$$

$$= \{ a_0 . a_1 a_2 \cdots \mid a_i \in \{0,1,2\} \}.$$

$$\mathbb{F}_3((t)) = \left\{ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \;\middle|\; \begin{matrix} \ell \in \mathbb{Z} \\ a_i \in \{0,1,2\} \end{matrix} \right\}$$

$$= \left\{ a_{-\ell} a_{-\ell+1} \cdots a_{-2} a_{-1} a_0 . a_1 a_2 \cdots \;\middle|\; \begin{matrix} \ell \in \mathbb{Z} \\ a_i \in \{0,1,2\} \end{matrix} \right\}$$