04.03.2025
ART Seminar
A. Ram

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Beautiful gems from Calculus

(2) What is an expression?

Examples of derivatives of expressions:

$$\frac{d}{dx}\left(\frac{x^2+6x+7}{3x+1}\right) \qquad \frac{d}{dx}(\tan(x)) \qquad \frac{d}{dx}\left(e^{3x+1}\sin(3x)\right)$$

$$\frac{d}{dx}(x^2\log(x)) \qquad \frac{d}{dx}\left(\cos(x)^{\sin(x)}\right)$$

Lets be honest with ourselves and our students. An expression is not a function

$$\mathbb{C}[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{C}\}$$

contains

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$$

An expression is an element of

$$\mathbb{C}((x)) = \left\{ \frac{f(x)}{g(x)} \mid f(x), g(x) \in \mathbb{C}[[x]] \text{ and } g(x) \neq 0 \right\}$$

with $\frac{a(x)}{b(x)} = \frac{c(x)}{d(x)}$ if $a(x)d(x) = b(x)c(x)$.

A more complicated structure than $\mathbb{C}[[x]]$ is

$$\mathbb{C}[x] = \left\{ a_0 + a_1x + a_2x^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{C} \text{ and there exists} \\ N \in \mathbb{Z}_{>0} \text{ such that if} \\ \ell \in \mathbb{Z}_{\geq N} \text{ then } a_\ell = 0 \end{array} \right\}$$

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① The exponential

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Note: $\cos(x)$, $\sin(x)$, $\cosh(x)$, $\sinh(x)$
are all fakes
since, with $i^2 = -1$,

$$\cos(x) = \cosh(ix), \qquad \sin(x) = i\sinh(ix)$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

Theorem If $xy = yx$ then
$$e^{x+y} = e^x e^y$$

Warning:

$$\sin^2(x) \neq \sin(x)^2$$

as this is equivalent to

$$\sin^{-1}(x) \neq \sin(x)^{-1}$$

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(3) Combining f and g

(a) $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

(b) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$

(c) $\frac{d}{dx}(f\circ g) = \frac{df}{dg}\cdot\frac{dg}{dx}$

(d) $\frac{d}{dx}(f^g) = gf^{g-1}\frac{df}{dx} + f^g\log(f)\frac{dg}{dx}$

Note: (c) and (d) follow from (a) and (b).

A <u>constant</u> is an element of

$$\ker\left(\frac{d}{dx}\right) = \left\{ f\in\mathbb{C}((x)) \;\middle|\; \frac{df}{dx}=0 \right\} = \mathbb{C}$$

Since $\frac{d}{dx}:\mathbb{C}((x))\longrightarrow\mathbb{C}((x))$ then

$$\int x^4\,dx = \left(\frac{d}{dx}\right)^{-1}(x^4) = \frac{1}{5}x^5 + \ker\left(\frac{d}{dx}\right) = \frac{1}{5}x^5+\mathbb{C}$$

Main topics in Calculus

ALGEBRA and GRAPHING

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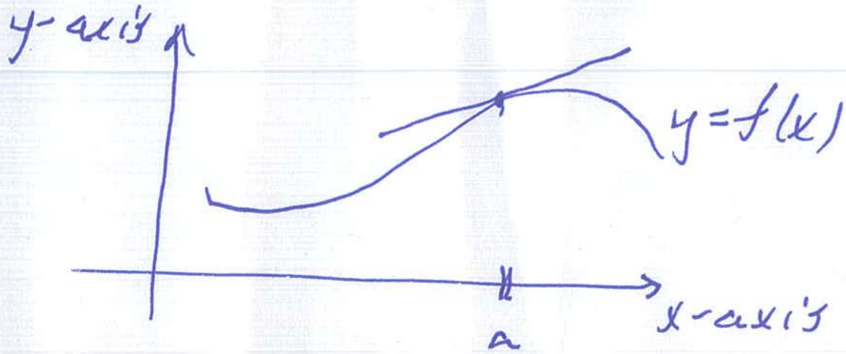
(4)

(4) Graphing Theorem I

(The theorem of rates)

$$ev_a\left(\frac{df}{dx}\right) = \left(\begin{matrix}\text{slope of graph of } f(x)\\ \text{at } x=a\end{matrix}\right)$$



(5) Graphing theorem II

(The theorem of a little at a time)

Let
$$A(x) = \left(\begin{matrix}\text{sum of a bunch of little things}\\ \text{where the } a^{th} \text{ little thing is } f(a)\end{matrix}\right)$$

Let $f(p)$ be the phirst little thing

Then
$$ev_a\left(\frac{dA}{dx}\right) = f(a)$$

Equivalently/notationally $\int_{z=p}^{z=x} f(z)\,dz = A(x)$