KZ lecture 4

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The plan

$$KZ: \{\mathbb{H}\text{-modules}\} \longrightarrow \{H_{fin}\text{-modules}\}$$
$$\Delta(E) \longmapsto E_q$$

Data:

$\mathfrak{a}^* = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$

W_{fin} a finite reflection group for $\mathfrak{a}^*$

$\mathcal{R} = \{\text{reflections } s\}$ $\quad \mathcal{A} = \{\text{hyperplanes } \mathfrak{a}^s\}$

$\mathfrak{a}^0 = \mathfrak{a}^* - \left(\bigcup_{\mathfrak{a}^s \in \mathcal{A}} \mathfrak{a}^s \right)$ with basepoint a_0

Step 1: Define $\mathbb{H}$

Add denominators to get $\mathbb{H}^0$

Convert to $\mathcal{D}^0$

Step 2: Let $E = \mathrm{span}\{e_1, \ldots, e_d\}$ be a W_{fin}-module

Define $\Delta(E)^0$, a $\mathcal{D}^0$-module

$HS(\Delta(E)^0) = \{p \in \Delta(E)^0 \mid \text{if } \lambda^\vee \in \mathfrak{a} \text{ then } \partial_{\lambda^\vee} p = 0\}$

Step 3 Let $f_1, \ldots, f_d \in HS(\Delta(E)^0)$ with $f_j(a_0) = e_j$.

Let $E_q = \mathrm{span}\{e_1, \ldots, e_d\}$.

For $s \in R$ define $T_s^{-1} \in \mathrm{End}(E_q)$ by

$$T_s^{-1} e_j = t_s^{-1} f_j(s a_0)$$

The matrices T_s make E_q into an H_{fin}-module

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General set up

Let $n \in \mathbb{Z}_{>0}$,

$\mathfrak{a}^* = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\mathfrak{a} = \mathrm{span}\{\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee\}$

and $\langle , \rangle : \mathfrak{a}^* \otimes \mathfrak{a} \to \mathbb{C}$ given by $\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}$.

Let W_{fin} be a subgroup of $GL(\mathfrak{a}^*)$ generated by

$$R = \{ s \in W_{fin} \mid s \text{ is a reflection} \}$$

The set of reflecting hyperplanes is

$\mathcal{A} = \{ \mathfrak{a}^s \mid s \in R \}$ where $\mathfrak{a}^s = \{ \mu \in \mathfrak{a}^* \mid s\mu = \mu \}$

Define an action of W_{fin} on $\mathfrak{a}$ by requiring

$\langle \mu, w\lambda^\vee \rangle = \langle w^{-1}\mu, \lambda^\vee \rangle$ for $w \in W_{fin}$, $\lambda^\vee \in \mathfrak{a}$, $\mu \in \mathfrak{a}^*$.

For $s \in R$ choose $\alpha_s \in \mathfrak{a}^*$ and $\alpha_s^\vee \in \mathfrak{a}$ so that

$$\mathfrak{a}^s = \{ \mu \in \mathfrak{a}^* \mid \langle \mu, \alpha_s^\vee \rangle = 0 \}.$$

$$s\mu = \mu - \langle \mu, \alpha_s^\vee \rangle \alpha_s, \quad \text{for } \mu \in \mathfrak{a}^*$$

$$s\lambda^\vee = \lambda^\vee - \langle \alpha_s, \lambda^\vee \rangle \alpha_s^\vee, \quad \text{for } \lambda^\vee \in \mathfrak{a}.$$

The <u>configuration space</u> is

$$\mathfrak{a}^\circ = \mathfrak{a}^* - \Big(\bigcup_{\mathfrak{a}^s \in \mathcal{A}} \mathfrak{a}^s \Big).$$

Fix $a_0 \in \mathfrak{a}^\circ$ (a basepoint).

Conversion $\tilde{H}^\circ$ to $\mathcal{D}^\circ$

$\tilde{H}^\circ$ has generators $\Delta, \frac{1}{\Delta}, x_\mu, t_w, y_{\lambda^\vee}$
and relations

$$x_{\mu+\nu} = x_\mu + x_\nu, \quad x_{c\mu} = c x_\mu \quad x_\mu x_\nu = x_\nu x_\mu \qquad (1)$$

$$\frac{1}{\Delta}\Delta = \Delta \cdot \frac{1}{\Delta} = 1 \quad \text{and} \quad \Delta = \prod_{s \in R} x_{\alpha_s} \qquad (2)$$

$$t_w t_v = t_{wv}, \quad t_w x_\mu = x_{w\mu} t_w \qquad (3)$$

$$y_{\lambda^\vee + \gamma^\vee} = y_{\lambda^\vee} + y_{\gamma^\vee}, \quad y_{c\lambda^\vee} = c y_{\lambda^\vee}, \quad y_{\lambda^\vee} y_{\gamma^\vee} = y_{\gamma^\vee} y_{\lambda^\vee} \qquad (4)$$

$$t_w y_{\lambda^\vee} = y_{w\lambda^\vee} t_w \quad \text{and} \quad y_{\lambda^\vee} x_\mu = x_\mu y_{\lambda^\vee} + \kappa\langle \mu, \lambda^\vee\rangle + \sum_{s \in R} c_s \langle \mu, \alpha_s^\vee\rangle\langle \alpha_s, \lambda^\vee\rangle t_s \qquad (5)$$

$\mathcal{D}^\circ$ has generators $\Delta, \frac{1}{\Delta}, x_\mu, t_w, \partial_{\lambda^\vee}$
with relations (1), (2), (3)

$$\partial_{\lambda^\vee + \gamma^\vee} = \partial_{\lambda^\vee} + \partial_{\gamma^\vee}, \quad \partial_{c\lambda^\vee} = c\,\partial_{\lambda^\vee}, \quad \partial_{\lambda^\vee}\partial_{\gamma^\vee} = \partial_{\gamma^\vee}\partial_{\lambda^\vee} \qquad (4')$$

$$t_w \partial_{\lambda^\vee} = \partial_{w\lambda^\vee} t_w \quad \text{and} \quad \partial_{\lambda^\vee} x_\mu = x_\mu \partial_{\lambda^\vee} + \langle \mu, \lambda^\vee\rangle. \qquad (5')$$

Proposition (a) [Gri06, Prop 2.3]. If $\lambda^\vee \in \mathfrak{a}$ and $f \in S(\mathfrak{a}^*)$ then

$$y_{\lambda^\vee} f = f y_{\lambda^\vee} + \kappa \frac{\partial f}{\partial \lambda^\vee} - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee\rangle \left(\frac{f - sf}{x_{\alpha_s}}\right) t_s$$

(b) If $\kappa \neq 0$ then $\tilde{H}^\circ \cong \mathcal{D}^\circ$ via the conversion

$$y_{\lambda^\vee} = \kappa \partial_{\lambda^\vee} - \sum_{s \in R} c_s \langle \alpha_s, \lambda^\vee\rangle \frac{1}{x_{\alpha_s}}(1 - t_s).$$

The $\mathcal{D}^\circ$-module $\Delta(E)^\circ$

Let $E = \mathrm{span}\{e_1, \dots, e_d\}$ be a W_{fin}-module and define $(t_s)_{ij} \in \mathbb{C}$ by

$$t_s e_j = \sum_{i=1}^{d} (t_s)_{ij} e_i.$$

Let $\Delta(E)^\circ$ be the H°-module generated by E with

$$y_{\lambda^\vee} e_j = 0 \quad \text{for } \lambda^\vee \in \mathfrak{a} \text{ and } j \in \{1, \dots, d\}.$$

Then

$$\Delta(E)^\circ = \{p = p_1 e_1 + \cdots + p_d e_d \mid p_j \in \mathcal{O}_{\mathfrak{a}^\circ}\}$$

The space of <u>horizontal sections</u> is

$$HS(\Delta(E)^\circ) = \left\{ p \in \Delta(E)^\circ \;\middle|\; \begin{array}{l} \text{if } \lambda^\vee \in \mathfrak{a} \text{ then} \\ \partial_{\lambda^\vee} p = 0 \end{array} \right\}$$

<u>Theorem</u> Assume $\kappa \neq 0$. Then $p = p_1 e_1 + \cdots + p_d e_d$ is a horizontal section if and only if

$$\frac{\partial p_i}{\partial x_k} = \sum_{s \in R}{}' c_s \langle \alpha_s, \varepsilon_k^\vee \rangle \frac{1}{x_{\alpha_s}} \left(-p_i + \sum_{j=1}^{d} (t_s)_{ij} p_j \right)$$

for $i \in \{1, \dots, d\}$ and $k \in \{1, \dots, n\}$.

Proof Let $p = p_1 e_1 + \cdots + p_d e_d \in \Delta(E)^0$. Then

$$y_{\lambda^\vee} p = \left(\sum_{j=1}^{d} p_j y_{\lambda^\vee} e_j\right) + \kappa\left(\sum_{j=1}^{d} \frac{\partial p_j}{\partial \lambda^\vee} e_j\right) - \sum_{s\in R}\sum_{j=1}^{d} c_s \langle \alpha_s, \lambda^\vee\rangle \left(\frac{p_j - s p_j}{X_{\alpha_s}}\right) t_s e_j$$

$$= \kappa\left(\sum_{j=1}^{d} \frac{\partial p_j}{\partial\lambda^\vee} e_j\right) - \sum_{s\in R}\sum_{j=1}^{d} c_s\langle\alpha_s,\lambda^\vee\rangle \frac{1}{X_{\alpha_s}}\left(p_j t_s e_j - (s p_j)\, t_s e_j\right)$$

and

$$y_{\lambda^\vee} p = \kappa\, \partial_{\lambda^\vee} p - \sum_{s\in R} c_s\langle\alpha_s,\lambda^\vee\rangle \frac{1}{X_{\alpha_s}}(1-t_s)p$$

$$= \kappa\, \partial_{\lambda^\vee} p - \sum_{s\in R} c_s \langle\alpha_s,\lambda^\vee\rangle\frac{1}{X_{\alpha_s}}\sum_{j=1}^{d} \left(p_j e_j - (s p_j)\, t_s e_j\right)$$

Assuming $\kappa\neq 0$ then $p\in HS(\Delta(E)^0)$ if and only if

$$\kappa\left(\sum_{j=1}^{d} \frac{\partial p_j}{\partial\lambda^\vee} e_j\right) = \sum_{s\in R}\sum_{j=1}^{d} c_s\langle\alpha_s,\lambda^\vee\rangle \frac{1}{X_{\alpha_s}}\left(\begin{array}{l} p_j t_s e_j - \cancel{(s p_j)\, t_s e_j} \\ - p_j e_j + \cancel{(s p_j)\, t_s e_j}\end{array}\right)$$

Putting $\lambda^\vee = \varepsilon_k^\vee$ and taking coefficients of e_i on each side gives

$$\kappa\frac{\partial p_i}{\partial x_k} = \sum_{s\in R} c_s\langle\alpha_s,\varepsilon_k^\vee\rangle\frac{1}{X_{\alpha_s}}\left(-p_i + \sum_{j=1}^{d} (t_s)_{ij}\, p_j\right)$$

for $i\in\{1,\dots,d\}$ and $k\in\{1,\dots,n\}$. //

The group $G(1,1,3)$

$$\mathfrak{a}^* = \mathrm{span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \qquad \mathfrak{a} = \mathrm{span}\{\varepsilon_1^\vee, \varepsilon_2^\vee, \varepsilon_3^\vee\}.$$

with $\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}$. With respect to the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$,

$$W_{\mathfrak{a}^*} = \{1, s_0s_1, s_1s_0, s_0, s_1, s_2\} \quad \text{with}$$

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad s_0s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad s_1s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 \qquad \alpha_2 = \varepsilon_2 - \varepsilon_3 \qquad \alpha_0 = \varepsilon_1 - \varepsilon_3$$

$$\alpha_1^\vee = \varepsilon_1^\vee - \varepsilon_2^\vee \qquad \alpha_2^\vee = \varepsilon_2^\vee - \varepsilon_3^\vee \qquad \alpha_0^\vee = \varepsilon_1^\vee - \varepsilon_3^\vee$$

Then

$$\mathfrak{a}^\circ = \left\{ \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \,\middle|\, \begin{matrix} \mu_1 \neq \mu_2 \\ \mu_1 \neq \mu_3 \\ \mu_2 \neq \mu_3 \end{matrix} \right\} \quad \text{with base point } a_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and

$$R = \{s_1, s_2, s_0\} \quad \text{and} \quad \mathcal{A} = \{\mathfrak{a}^{\alpha_1^\vee}, \mathfrak{a}^{\alpha_2^\vee}, \mathfrak{a}^{\alpha_0^\vee}\}.$$

Let $\kappa, c_0 \in \mathbb{C}$. <u>The x's and y's</u>

$$x_1 = x_{\varepsilon_1}, \qquad y_1 = y_{\varepsilon_1^\vee} \qquad x_{\mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3} = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3$$

$$x_2 = x_{\varepsilon_2}, \qquad y_2 = y_{\varepsilon_2^\vee}$$

$$x_3 = x_{\varepsilon_3}, \qquad y_3 = y_{\varepsilon_3^\vee} \qquad y_{\lambda_1\varepsilon_1^\vee + \lambda_2\varepsilon_2^\vee + \lambda_3\varepsilon_3^\vee} = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3.$$

$$x_i x_j = x_j x_i \qquad y_i y_j = y_j y_i \qquad \partial_i \partial_j = \partial_j \partial_i.$$

For $G(1,1,3)$

Let $E = \mathrm{span}\{e_1, e_2\}$ write W_{fin}-action given by

$$t_{s_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_{s_2} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad t_{s_0} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

with respect to the basis $\{e_1, e_2\}$.

The KZ equations for $HS(\Delta|E]^0)$ are

$$\frac{\partial p_1}{\partial x_1} = \frac{\partial p_1}{\partial \zeta_1^\vee} = \frac{c_0}{\kappa}\left(\frac{1}{x_1 - x_2}(-p_1 + p_1) + \frac{1}{x_1 - x_3}\left(-p_1 + \frac{1}{2}p_1 - \frac{3}{2}p_2\right)\right)$$

$$\frac{\partial p_1}{\partial x_2} = \frac{c_0}{\kappa}\left(\frac{-1}{x_1 - x_2}(-p_1 + p_1) + \frac{1}{x_1 - x_3}\left(-p_1 + \frac{1}{2}p_1 + \frac{3}{2}p_2\right)\right)$$

$$\frac{\partial p_1}{\partial x_3} =$$

$$\frac{\partial p_2}{\partial x_1} =$$

$$\frac{\partial p_2}{\partial x_2} =$$

$$\frac{\partial p_2}{\partial x_3} =$$