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Flag varieties
Lecture 4

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Questions and further directions

(1) Generalized cohomology

(2) Chern-Schwartz-MacPherson classes and stable envelopes.

(3) Affine flag varieties

(4) Representations of affine and double affine Hecke algebras

(5) p-compact groups

References

(1) Ganter-Ram arXiv:1212.5742,

Harada-Henriques-Holm arXiv:0409305

(2) Aluffi-Mihalcea arXiv:1508.01535,

Maulik-Okounkov and Aganagic-Okounkov, see Okounkov's Park City lecture notes arXiv:1604.00423

Thesis of Changjiang Su at Columbia (from his web page)

(3) Parkinson-Ram-Schwer arXiv:0801.0709
Lusztig's ICM 90 article §11
Braverman-Finkelberg and Kato-Naito-Sagaki arXiv:1702.02408
(4) Kazhdan-Lusztig, Inventiones 1987
Garland-Grojnowski, Varagnolo-Vasserot and Oblomkov-Yun arXiv:1407.5685

(5) Omar Ortiz, thesis at Univ. of Melbourne and
J. Algebra **427** (2015) 426-454.

Generalized cohomology

Ordinary cohomology $H_T(pt) = \mathbb{C}[y_1, y_2, \ldots, y_n]$

K-theory $K_T(pt) = \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$

Elliptic cohomology $Ell_T(pt)$ is a ring of theta functions

Cobordism $\Omega_T(pt) = \mathbb{L}[[y_1, \ldots, y_n]]$ where $\mathbb{L}$ is the Lazard ring.

Project: Properly understand the Chevalley-Shephard-Todd theorem
i.e. why the Borel model is a free $H_T(pt)$-module,
in this context (see Serre, Bernstein-Schwarzmann, Looijenga,
Harada-Holm-Henriques and Ganter's Compositio paper)

Project: What are the Schubert classes in this context?

The push-pull operators give Bott-Samelson classes:

$D_j [Z_{i_1 \cdots i_\ell}] = [Z_{j i_1 \cdots i_\ell}]$ but $[Z_{121}] \neq [Z_{212}]$.

For Schubert classes we must have $[X_{s_1 s_2 s_1}] = [X_{s_2 s_1 s_2}]$.

Chern-Schwartz-MacPherson classes and Stable envelopes

The CSM classes in $H_T(X)$ are given by a universal property for constructible functions, by MacPherson's proof of a conjecture of Deligne-Grothendieck. Aluffi-Mihalcea (2015) proved that the CSM classes $[S_w]$ in $H_T(G/B)$ satisfy

$$(D_i - t_{s_i})[S_w] = [S_{s_i w}] \quad \text{if } \ell(s_i w) > \ell(w).$$

Projects: Use moment graphs to determine a_{uw}, b_{uv}^{w} and c_{uv}^{w} given by

$$[S_u][S_v] = \sum_w b_{uv}^{w}[S_w]$$

$$[X_w] = \sum_u a_{uw}[S_u],$$

$$[X_u][X_v] = \sum_w c_{uv}^{w}[X_w]$$

Another project: Does Aluffi-Mihalcea work in elliptic cohomology and/or cobordism?

Another project: Are the $[S_w]$ stable envelopes in the sense of Maulik-Okounkov? (See the 2016 Columbia University PhD Thesis of Changjiang Su.)

Affine flag varieties

Let $G = GL_n(\mathbb{C}[\epsilon,\epsilon^{-1}])$ or $GL_n(\mathbb{C}((\epsilon)))$ or $GL_n(\mathbb{Q}_p)$

$$I^+ = \left\{ (g_{ij}) \in GL_n(\mathbb{C}[\epsilon]) \;\middle|\; (g_{ij}(0)) \in \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \right\}$$

$$I^0 = \left\{ (g_{ij}) \in GL_n(\mathbb{C}[\epsilon,\epsilon^{-1}]) \;\middle|\; (g_{ij}) \in \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} \right\}$$

$$I^- = \left\{ (g_{ij}) \in GL_n(\mathbb{C}[\epsilon^{-1}]) \;\middle|\; (g_{ij}(\infty)) \in \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} \right\}$$

G/I^+ is the positive level affine flag variety (the thin affine flag variety)

G/I^0 is the level 0 affine flag variety (the semi-infinite flag variety)

G/I^- is the negative level affine flag varieties (the thick affine flag variety)

Project: Determine the moment graph, Schubert classes and products

$$[X_u][X_v] = \sum_w c_{uv}^w [X_w] \quad \text{in these cases.}$$

Representations of affine Hecke algebras and double affine Hecke algebras

The operators $L_1, L_2, \ldots, L_n$ and $D_1, D_2, \ldots, D_{n-1}$ make $H_T(G/B)$ into a module for the (nil) affine Hecke algebra.

In the case of the affine flag varieties $H_T(G/I^+)$, $H_T(G/I^0)$, $H_T(G/I^-)$ are (nil) double affine Hecke algebra (DAHA) modules.

Kazhdan-Lusztig Theorem

There are subvarieties $\mathcal{B}_{s,n}$ in G/B,

$\mathcal{B}_{s,n}$ = generalized Springer fiber

such that

(a) $H_T(\mathcal{B}_{s,n})$ are affine Hecke algebra modules.

(b) $H_T(\mathcal{B}_{s,n})$ has a unique simple quotient as an H-module.

(c) All simple H-modules are obtained this way.

The work of Oblomkov-Yun starts to deal with the affine flag variety cases.

Project: Use moment graphs to study the H-modules $H_T(\mathcal{B}_{s,n})$.