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## Flag varieties
## Lecture 3

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Summary to date:

$$\begin{array}{l} G = GL_n(\mathbb{C}) \\ \cup\!\!| \\ B = \{\text{upper triangular matrices}\} \\ \cup\!\!| \\ T = \{\text{diagonal matrices}\} \end{array}$$

$W = \{\text{permutation matrices}\}$

The <u>flag</u> <u>variety</u> is $G/B$ and $\quad G = \bigsqcup_{w\in W} BwB$.

Let $w \in W$ and $w = s_{i_1} \cdots s_{i_\ell}$ a reduced word.

The <u>Schubert</u> <u>cell</u> is $BwB = \{ y_{i_1}(c_1) y_{i_2}(c_2) \cdots y_{i_\ell}(c_\ell) B \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$

and the <u>Schubert</u> <u>variety</u> is $\overline{BwB} = \bigsqcup_{v \le w} BvB$

where $v \le w$ if $v$ is a subword of $w = s_{i_1} \cdots s_{i_\ell}$.

The moment graph of $G/B$ has

vertices: $w \in W$ $\qquad$ $H_T(pt) = \mathbb{C}[y_1, \dots, y_n]$. $\qquad$ Let $y_{ij} = y_i - y_j$.

labeled edges: $\underset{w}{\bullet}\overset{x_{ij}}{\text{———}}\underset{ws_{ij}}{\bullet}$ for $i<j$

## T-equivariant cohomology of flag varieties

$$H_T(G/B) = \{ (f_w)_{w \in W} \mid f_w \in H_T(pt) \text{ and } f_w - f_{s_{ij}w} \in y_{ij} H_T(pt) \}$$

Elements of $H_T(G/B)$ are tuples of polynomials

- one $f_w$ for each vertex
- a condition $f_w - f_{s_{ij} \cdot w}$ is divisible by $y_i - y_j$ for each edge $\underset{w}{\bullet}\overset{x_{ij}}{\text{———}}\underset{ws_{ij}}{\bullet}$

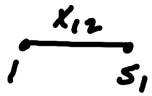
The product in $H_T(G/B)$ is pointwise, $\quad (fg)_w = f_w g_w$

$H_T(G/B)$ is an $H_T(pt)$-module, $\quad (y_i f)_w = y_i f_w$

and $H_T(G/B)$ is an $H_T(pt)$-algebra with identity $1$ given by $1_w = 1$.

The line bundles in $H_T(G/B)$ are $L_1, \dots, L_n$ in $H_T(G/B)$ given by

$$(L_i)_w = y_{w(i)}$$

$H_T(\mathbb{P}^1)$: $\mathbb{P}^1 = GL_2(\mathbb{C})/B$ $\quad H_T(pt) = \mathbb{C}[y_1, y_2]$ $\quad$ The moment graph is 


Some elements of $H_T(G/B)$:

$[X_1] =$ 

$L_1 =$ 

$y_1 =$ 


$[X_{s_1}] =$ 

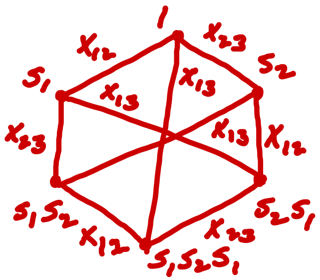
$L_2 =$ 

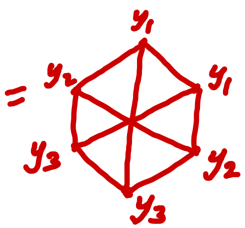
$y_2 =$ 

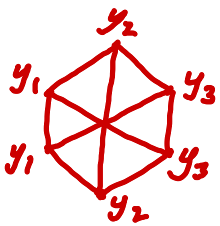

Note: $L_1 L_2 =$ 

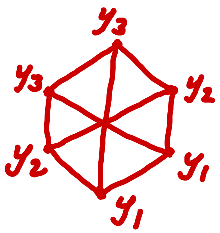
$= y_1 y_2$ and $L_1 + L_2 =$ 

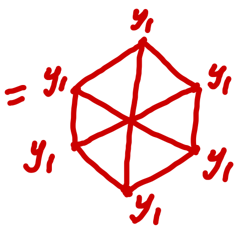
$= y_1 + y_2$

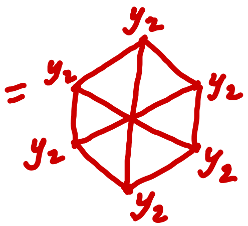
For $G = GL_3(\mathbb{C})$, $H_T(pt) = \mathbb{C}[y_1, y_2, y_3]$. The moment graph is 

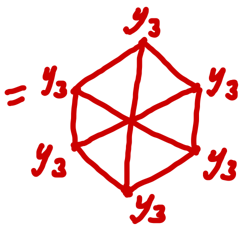

$L_1 =$ 

$L_2 =$ 

$L_3 =$ 


$y_1 =$ 

$y_2 =$ 

$y_3 =$ 

# The Borel model for $H_T(G/B)$

**Theorem** As $H_T(pt)$-algebras,

$$H_T(G/B) \cong \frac{\mathbb{C}[y_1, y_2, \ldots, y_n, L_1, L_2, \ldots, L_n]}{\langle f(L_1, L_2, \ldots, L_n) = f(y_1, y_2, \ldots, y_n) \mid f \in \mathbb{C}[y_1, \ldots, y_n]^{S_n} \rangle}$$

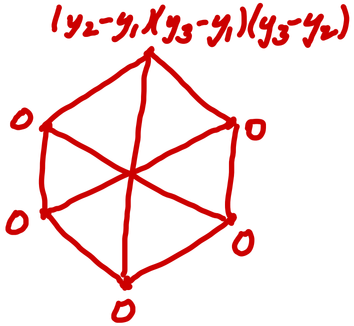
Where $\mathbb{C}[y_1, \ldots, y_n]^{S_n} = \left\{ f \in \mathbb{C}[y_1, \ldots, y_n] \;\middle|\; \begin{array}{l} \text{if } i \in \{1, \ldots, n\} \text{ then} \\ f(y_1, \ldots, y_{i+1}, y_i, \ldots, y_n) = f(y_1, \ldots, y_i, y_{i+1}, \ldots, y_n) \end{array} \right\}$

**Example**: For $G = GL_3(\mathbb{C})$

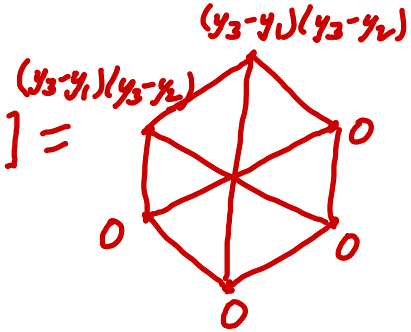
$$H_T(G/B) \cong \frac{\mathbb{C}[y_1, y_2, y_3, L_1, L_2, L_3]}{\left\langle \begin{array}{l} L_1 + L_2 + L_3 = y_1 + y_2 + y_3, \\ L_1L_2 + L_1L_3 + L_2L_3 = y_1y_2 + y_1y_3 + y_2y_3 \\ L_1L_2L_3 = y_1y_2y_3 \end{array} \right\rangle}$$

# The Schubert classes in $H_T(GL_3(\mathbb{C})/B)$

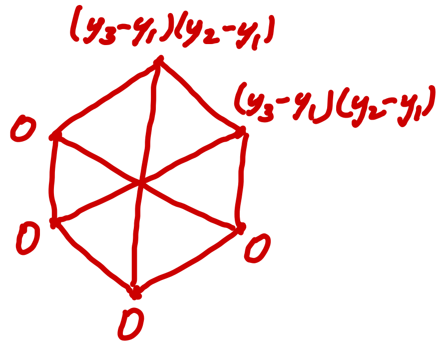
$[X_1] =$



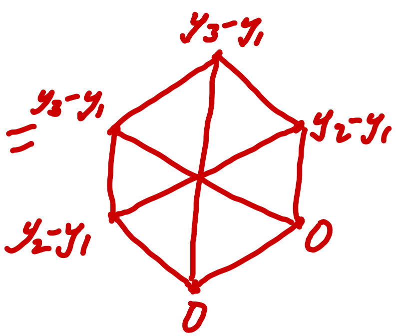

$[X_{s_1}] =$



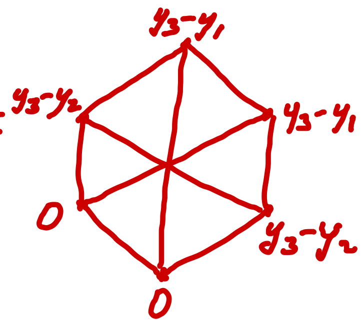

$[X_{s_2}] =$



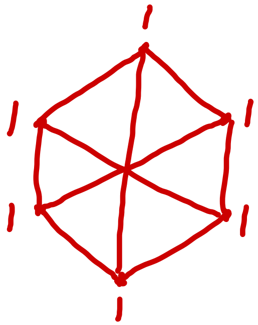

$[X_{s_1 s_2}] =$




$[X_{s_2 s_1}] =$




$[X_{s_1 s_2 s_1}] =$



## Schubert Classes $[X_w]$ for $w \in W$

The Schubert classes in $H_T(G/B)$ are $[X_w]$ given by

(a) (support) $[X_w]_v = 0$ unless $v \le w$,

(b) (smooth points) $[X_w]_w = \prod_{ws_{ij} < w} (y_i - y_j)$,

(c) (pinning) $[X_w]_v \in \mathbb{C}[y_1, y_2, \ldots, y_n]$ is homogeneous of degree

$$\mathrm{Card}\{(i,j) \mid i<j \text{ and } ws_{ij} < w\}$$

For $i \in \{1, 2, \ldots, n-1\}$ let $L_{-\alpha_i} = L_{i+1} - L_i$

and $t_{s_i}: H_T(G/B) \to H_T(G/B)$ given by $(t_{s_i} f)_w = f_{s_i w}$

The push-pull operators are $D_i : H_T(G/B) \to H_T(G/B)$ given by

$$D_i = (1 + t_{s_i}) \frac{1}{L_{-\alpha_i}}$$

## Theorem

(a) $H_T(G/B)$ is a free $H_T(pt)$-module with basis $\{[X_w] \mid w \in W\}$.

(b) $[X_1]$ is given by $[X_1]_w = \begin{cases} \prod_{i<j} (y_j - y_i), & \text{if } w = 1, \\ 0, & \text{if } w \neq 1. \end{cases}$

and $\quad D_i [X_w] = \begin{cases} [X_{s_i w}], & \text{if } \ell(s_i w) > \ell(w), \\ 0, & \text{if } \ell(s_i w) < \ell(w) \end{cases}$

This process inductively determines the $[X_w]$.

**Proposition** As operators on $H_T(G/B)$, $D_1, \dots, D_{n-1}, L_1, \dots, L_n$ satisfy

$$L_i L_j = L_j L_i, \quad D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}, \quad D_i^2 = 0, \quad D_k D_j = D_j D_k \text{ if } k \neq j \pm 1,$$

$$D_i L_i = L_{i+1} D_i + 1, \quad D_i L_{i+1} = L_i D_i - 1, \quad D_i L_j = L_j D_i \text{ if } j \notin \{i, i+1\}$$

Hence $H_T(G/B)$ is a module for the (nil) affine Hecke algebra.