Univ. of Melbourne   Number Theory seminar 09.03.2015

Line bundles

Work in the category of complex analytic spaces.

A line bundle on X is a surjective morphism

$$\mathbb{L} \xrightarrow{p} X \qquad \text{such that}$$

(a) if $x \in X$ then $p^{-1}(x) = \mathbb{C}$ as a $\mathbb{C}$-vector space,

(b) if $x \in X$ then there exists a neighborhood $\mathcal{U}$ of x and an isomorphism $\varphi_u : \mathbb{C} \times \mathcal{U} \to \varphi^{-1}(\mathcal{U})$ such that

(ba) $p(\varphi_u(z, x)) = x$, and

(bb) If $y \in \mathcal{U}$ then $\varphi_u\big|_{\mathbb{C}\times y} : \mathbb{C} \times \{y\} \to p^{-1}(y)$ is a linear transformation.

The space of sections of $\mathbb{L}$ is

$$\Gamma(X, \mathbb{L}) = \{ s : X \to \mathbb{L} \mid p \circ s = id_X \}.$$

An automorphism of $\mathbb{L}$ is a pair of morphisms (φ, ψ) with

$$\begin{tikzcd} \mathbb{L} \arrow[r, "\psi"] \arrow[d, "p"'] & \mathbb{L} \arrow[d, "p"] \\ X \arrow[r, "\varphi"'] & X \end{tikzcd}$$

commutes and if $x \in X$ then $\psi : p^{-1}(x) \to p^{-1}(\phi(x))$ is a $\mathbb{C}$-linear transformation.

Under composition these form the group $\mathrm{Aut}(\mathbb{L})$.
$\mathrm{Aut}(\mathbb{L})$ acts on $\Gamma(X, \mathbb{L})$ by

$$\mathcal{U}_{(\phi,\psi)} = \psi \circ s \circ \phi^{-1}.$$

Holomorphic functions on X

The trivial line bundle on X is
$$\begin{aligned} \mathbb{C}\times X &\longrightarrow X \\ (z,x) &\longmapsto x. \end{aligned}$$

Since a section of $\mathbb{C}\times X \to X$ is
$$\begin{aligned} s: X &\longrightarrow \mathbb{C}\times X \\ x &\longmapsto (h(x),x) \end{aligned} \quad \text{with} \quad \begin{aligned} h: X &\longrightarrow \mathbb{C} \\ x &\longmapsto h(x) \end{aligned} \quad \text{holomorphic}$$

then $\Gamma(X, \mathbb{C}\times X) = \{\text{holomorphic functions } h: X \to \mathbb{C}\}$

$Heis(2g,\mathbb{R})$ acts on $\Gamma(X,\mathbb{C}\times X)$ for $X=\mathbb{C}^g$

Fix
$$\tau \in \mathfrak{S}_g \quad \text{and} \quad \begin{aligned} \mathbb{R}^{2g} &\overset{\sim}{\longrightarrow} \mathbb{C}^g \\ (y_1,y_2) &\longmapsto \tau y_1 + y_2 \end{aligned}$$

Let $X=\mathbb{C}^g$, $\mathbb{L} = \mathbb{C}\times\mathbb{C}^g$ the trivial line bundle on X and
$$\begin{aligned} Heis(2g,\mathbb{R}) &\longrightarrow Aut(\mathbb{C}\times X \xrightarrow{p} X) \\ (\lambda, y_1, y_2) &\longmapsto (\varphi_{(\lambda,y_1,y_2)}, \psi_{(\lambda,y_1,y_2)}) \end{aligned}$$
by
$$\varphi_{(\lambda,y_1,y_2)}(z) = z - y, \quad \text{where } y = \tau y_1 + y_2$$
$$\psi_{(\lambda,y_1,y_2)}(\alpha, z) = (\alpha\lambda^{-1} e^{i\pi y_1^t(2z-y)}, z-y).$$

Then, on $\Gamma(X, \mathbb{C}\times X)$,
$$U_{(\lambda,y_1,y_2)}(f(z),z) = (f(z+y)\lambda^{-1}e^{i\pi y_1^t(2z+y)}, z)$$
which coincides with the action on Fock space
$$\mathcal{H}^2_{\theta}(\mathbb{C}^g,\tau) = \left\{ f:\mathbb{C}^g \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic and} \\ \int_{\mathbb{C}^g} |f(z)|^2 e^{-2\pi x_1^t \operatorname{Im}\tau x_1} dx_1 dx_2 \end{array} \right\}$$

Theta functions

Let $\sigma : \mathbb{Z}^{2g} \to \mathrm{Heis}(2g, \mathbb{R})$ be given by

$$\sigma(n_1, n_2) = (e^{2\pi i\, {}^t n_1 n_2}, n_1, n_2)$$

Using σ, $\mathbb{Z}^{2g}$ acts on the trivial line bundle $\mathbb{C} \times \mathbb{C}^g$ and on $\mathcal{H}_\theta^2(\mathbb{C}^g, \tau)$.

The theta function $\theta(\underline{x}, \tau)$ is the unique (up to normalization)

$$\theta(\underline{x}, \tau) \in \mathcal{H}_\theta^2(\mathbb{C}^g, \tau)^{\sigma(\mathbb{Z}^{2g})}$$

Notation: $e_{\mathbb{Z}} = e\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = \theta(\underline{x}, \tau)$.

$$e\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] = \theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\underline{x}, \tau) = U_{(e^{-i\pi\, {}^t a b}, a, b)}\, e\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right].$$

These are the theta functions with characteristic.

The quotient line bundle

$$L \subseteq \mathbb{Z}^{2g} \text{ a lattice}, \qquad X_{\tau, L} = \mathbb{C}^g / L$$

$\mathbb{L} = \mathbb{C} \times \mathbb{C}^g / L \longrightarrow \mathbb{C}^g / L$ the quotient of $\mathbb{C} \times \mathbb{C}^g \to \mathbb{C}^g$ by L.

Theorem Let $\{(a_1, b_1), \ldots, (a_d, b_d)\}$ be coset reps of $L^\perp / \mathbb{Z}^{2g}$. Then

$$\Gamma(X_{\tau, L}, \mathbb{L}) = \left(\mathcal{H}_\theta^2(\mathbb{C}^g, \tau)_{-\infty}\right)^{\sigma L}$$

has basis

$$\left\{ \theta\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right], \ldots, \theta\left[\begin{smallmatrix}a_d\\b_d\end{smallmatrix}\right] \right\}.$$

The group $\mathcal{G}(\mathbb{L})$

$$\sigma : \mathbb{Z}^{2g} \to \mathrm{Heis}(2g, \mathbb{R})$$
$$(n_1, n_2) \mapsto \left(e^{2\pi i \frac{1}{2} n_1^t n_2}, n_1, n_2\right) \quad \text{and} \quad L \subseteq \mathbb{Z}^{2g}$$

Let

$N(\sigma L)$ be the normalizer of σL in $\mathrm{Heis}(2g, \mathbb{R})$.

Then (Mumford, Proposition 3.1)

isomorphism? I'm confused ?

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}_1^\times \arrow[r] \arrow[d, dashed] & \frac{N(\sigma L)}{\sigma L} \arrow[r] \arrow[d, "s"] & \frac{L^\perp}{L} \arrow[r] \arrow[d, "s"] & 0 \\
1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \mathcal{G}(\mathbb{L}) \arrow[r] & K(\mathbb{L}) \arrow[r] & 0
\end{tikzcd}$$

where

$$L^\perp = \{ y \in \mathbb{R}^{2g} \mid A(y, l) \in \mathbb{Z}, \text{ for } l \in L \}$$

$$\mathcal{G}(\mathbb{L}) = \left\{ (\phi, \psi) \in \mathrm{Aut}(\mathbb{L}) \;\middle|\; \begin{matrix} \text{there exists } a \in \mathbb{C}^g \text{ with} \\ \phi(x) = x + a \end{matrix} \right\}$$

$$K(\mathbb{L}) = \left\{ a \in \mathbb{C}^g \;\middle|\; \begin{matrix} \text{there exists } (\phi, \psi) \in \mathcal{G}(\mathbb{L}) \text{ with} \\ \phi(x) = x + a. \end{matrix} \right\}$$

and $L^\perp / L$ is a finite abelian group.

$\Gamma(X_{\mathbb{L}, L}, \mathbb{L})$ is the irreducible Heisenberg representation of $\mathcal{G}(\mathbb{L})$.

$$d = \mathrm{Card}(L^\perp / \mathbb{Z}^{2g}) = \dim(\Gamma(X_{\mathbb{L}, L}, \mathbb{L})) = \frac{1}{g!}(c_1(\mathbb{L})^g),$$

where $c_1(\mathbb{L})$ is the first Chern class of $\mathbb{L}$.

$$\mathrm{Card}(K(\mathbb{L})) = d^2.$$

Embedding $X_{\tau,L}$ in projective space

The Lefschetz-Kodaira embedding theorem gives that

$$\phi_{\tau,L} : X_{\tau,L} \longrightarrow \mathbb{P}(\Gamma(X_{\tau,L}, \mathbb{L})) = \mathbb{P}^{d-1}$$

is an embedding.

Examples of descriptions of $Im(\phi_{\tau,L})$

(a) Mumford Tata Lectures I: $g=1$, $L = 2\cdot\mathbb{Z}^2$

(b) $g=1$, $L = 2\mathbb{Z}+\mathbb{Z}$ gives a representation of elliptic curves as double covers of $\mathbb{P}^1$ ramified in 4 points $\pm a, \pm a^{-1}$

(c) $g=1$, $L = 3\mathbb{Z}+\mathbb{Z}$ gives a representation of elliptic curves as cubic curves $X_0^3 + X_1^3 + X_2^3 + \lambda X_0 X_1 X_2 = 0$.

(d) $g=2$, $L = 2\mathbb{Z}^2 + \mathbb{Z}^2$ gives a representation of principally polarized 2-dim. abelian surfaces as double covers of "Kummer" quadratic surfaces with 16 nodes.

(e) $g=2$, $L = 4\mathbb{Z}+\mathbb{Z}^3$ gives a class of octic surfaces in $\mathbb{P}^3$.

(f) $g=2$, $L = 5\mathbb{Z}+\mathbb{Z}^3$ gives an "interesting story in $\mathbb{P}^4$"

(Horrocks and Mumford, Topology $\underline{12}$ (1973)).