Talk in "Coxeter groups seminar" 11.04.2014

Artin groups and affine Artin groups for Coxeter groups

Let (W,S) with $S=\{s_1,\ldots,s_\ell\}$ be a Coxeter group:

$$s_i^2=1 \quad\text{and}\quad \underbrace{s_is_js_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{s_js_is_j\cdots}_{m_{ij}\text{ factors}}$$

Let $\mathbb{V}$ be the reflection representation

$$s_i: \mathbb{V}\to\mathbb{V} \quad\text{given by}\quad s_i\lambda = \lambda - \langle\lambda,\alpha_i^\vee\rangle\alpha_i$$

R^+ an index set for reflections in W and

$$\mathfrak{h}^{\alpha^\vee} = \mathfrak{h}^{s_\alpha} = \{x\in\mathbb{V} \mid s_\alpha x = x\} = \{x\in\mathbb{V} \mid \langle x,\alpha^\vee\rangle = 0\}.$$

The <u>chamber</u> is $C=\{x\in\mathbb{V} \mid \langle x,\alpha^\vee\rangle > 0 \text{ for } \alpha\in R^+\}$

and the <u>Tits</u> <u>cone</u> is

$$\mathcal{I} = \bigcup_{w\in W} w(\overline{C}) \quad\text{and } \mathcal{I}^\circ \text{ is the interior of } \mathcal{I}.$$

The Artin group of W

$$Y = V + iI^{\circ} - \bigcup_{\alpha \in R^+} \mathfrak{h}^{\alpha^\vee} + i\mathfrak{h}^{\alpha^\vee}$$

with W-action given by

$$w(x+iy) = wx + i(wy), \quad \text{for } w \in W.$$

Let $c \in C$ and $* = ic$,
the basepoint of Y.

Theorem $\pi_1(Y/W)$ is generated by $T_1, \ldots, T_\ell$ with relations

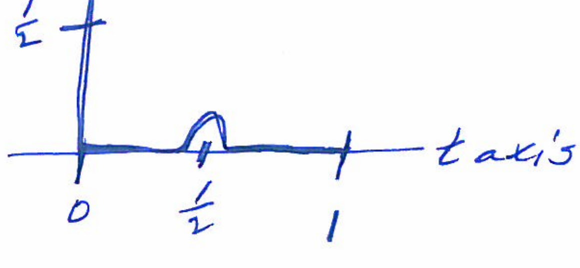
$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad \text{for } i \neq j.$$

where

$$T_j : [0,1] \to Y \to Y/W \quad \text{is given by}$$

$$T_j(t) = p(t) + i\big((1-t)c + t(s_j c)\big)$$

with $p : [0,1] \to [0, \tfrac{1}{2}]$ having graph



The affine Weyl group of W

$$\{\alpha_1,\dots,\alpha_\ell\} \subseteq \mathfrak{h}_{\mathbb{R}}^* \quad \text{and} \quad \{\alpha_1^\vee,\dots,\alpha_\ell^\vee\} \subseteq \mathfrak{h}_{\mathbb{R}}$$

$\dim(\mathfrak{h}_{\mathbb{R}}^*) \geq \ell + \mathrm{corank}(N)$ where $N = (\alpha_i^\vee(\alpha_j))_{1\le i,j\le \ell}$.

and W acts on $\mathfrak{h}_{\mathbb{R}}^*$ by

$$s_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee\rangle \alpha_i \quad \text{for } i = 1,2,\dots,\ell.$$

Let

$$Y = \mathfrak{h}_{\mathbb{R}}^* + i\mathcal{I}^\circ - \bigcup_{\substack{\alpha\in R^+\\ m\in\mathbb{Z}}} \mathfrak{h}^{\alpha^\vee,m} + i\,\mathfrak{h}^{\alpha^\vee}$$

where

$$\mathfrak{h}^{\alpha^\vee,m} = \{x\in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x,\alpha^\vee\rangle = m\}.$$

Let

$$Q = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i \quad \text{and} \quad \tilde{W} = W \ltimes Q = \{wt_\mu \mid w\in W, \mu\in Q\}$$

with

$$t_\mu t_\nu = t_{\mu+\nu} \quad \text{and} \quad wt_\mu = t_{w\mu}w.$$

$\tilde{W}$ acts on Y by

$$w(x+iy) = wx + i(wy) \quad \text{and} \quad t_\mu(x+iy) = (\mu+x)+iy.$$

Let $c\in C$ and $* = ic$

be the basepoint of Y.

where $C = \{x\in\mathfrak{h}_{\mathbb{R}}^* \mid \langle x,\alpha^\vee\rangle \in\mathbb{R}_{>0} \text{ for } \alpha\in R^+\}$.

<u>Theorem</u> $\pi_1(\tilde{Y}/\tilde{W})$ is generated by

$$T_1, \ldots, T_\ell \quad \text{and} \quad X^{\alpha_1}, \ldots, X^{\alpha_\ell}$$

with relations

$$X^{\alpha_i} X^{\alpha_j} = X^{\alpha_j} X^{\alpha_i} \quad \text{and} \quad \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad \text{for } i \neq j$$

$$T_i X^{\alpha_j} = \begin{cases} X^{\alpha_j} X^{r\alpha_i} T_i X^{-r\alpha_i}, & \text{if } -\langle \alpha_i^\vee, \alpha_j \rangle = 2r, \\ X^{\alpha_j} X^{(r+1)\alpha_i} T_i X^{-r\alpha_i}, & \text{if } -\langle \alpha_i^\vee, \alpha_j \rangle = 2r+1. \end{cases}$$

where

$$X^{\alpha_j} : [0,1] \to Y \quad \text{and} \quad T_j : [0,1] \to Y$$

given by

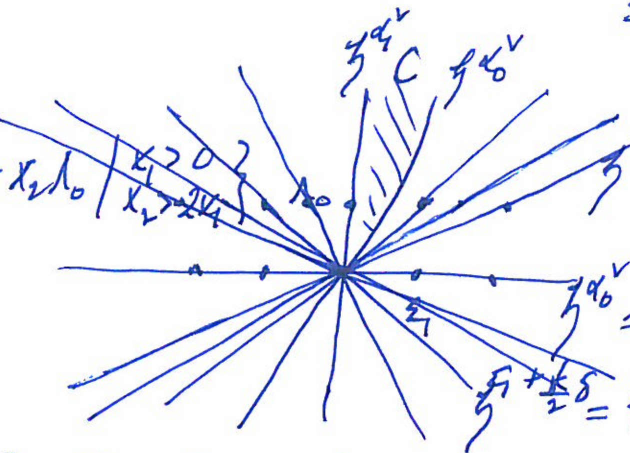
$$X^{\alpha_j}(t) = t\alpha_j + ic, \quad \text{and}$$

$$T_j(t) = p(t)\alpha_j + i((1-t)c + ts_j(c)).$$

<u>Example</u> $N = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ so $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$

$\dim(\mathbb{V}) = l = 2$ and

$\mathbb{V} = \mathrm{span}\{\Lambda_0, \varepsilon_1\}$ with $s_0\Lambda_0 = \Lambda_0 + \varepsilon_1$, $s_0\varepsilon_1 = -\varepsilon_1$
$s_1\Lambda_0 = \Lambda_0$, $s_1\varepsilon_1 = -\varepsilon_1$



$C = \left\{ x_1\varepsilon_1 + x_2\Lambda_0 \,\middle|\, \begin{matrix} x_1 > 0 \\ x_2 > 2x_1 \end{matrix} \right\}$

$\mathfrak{h}^{\alpha_1^\vee} = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_1 = 0 \} = \mathbb{R}\Lambda_0$

$\mathfrak{h}^{\alpha_0^\vee} = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_2 = 2x_1 \}.$

$\mathfrak{h}^{\varepsilon_1 + \frac{k}{2}\delta} = \{ x_1\varepsilon_1 + x_2\Lambda_0 \mid x_1 = \frac{k}{2} \cancel{x_2} \}$

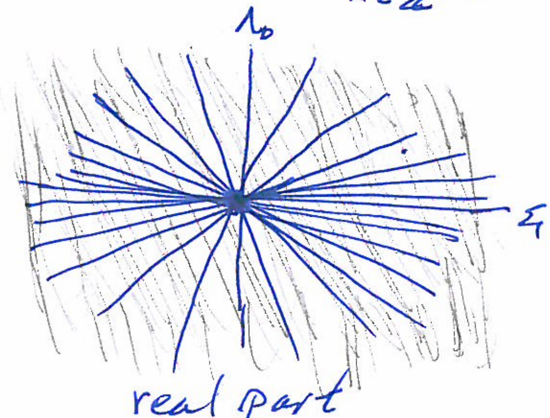
$\mathcal{I}^\circ = \mathcal{I} = \mathbb{R}\varepsilon_1 + \mathbb{R}_{>0}\Lambda_0$, $R^+ = \{ \varepsilon_1 + \frac{k}{2}\delta \mid k \in \mathbb{Z} \}$

Then

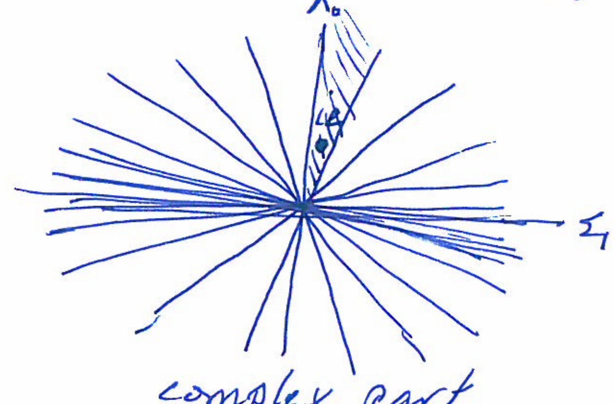
$$Y = \mathbb{V} + i\mathcal{I}^\circ - \bigcup_{\alpha \in R^+} \mathfrak{h}^{\alpha^\vee} + i\mathfrak{h}^{\alpha^\vee}$$

$$= (\mathbb{R}\varepsilon_1 + \mathbb{R}\Lambda_0) + i(\mathbb{R}\varepsilon_1 + \mathbb{R}_{>0})\Lambda_0 - \bigcup_{\alpha \in R^+} \mathfrak{h}^{\alpha^\vee} + i\mathfrak{h}^{\alpha^\vee}$$

$$= \mathbb{C}\varepsilon_1 + \mathbb{C}_1\Lambda_0 - \bigcup_{k \in \mathbb{Z}} \left\{ x_1\varepsilon_1 + x_2\Lambda_0 + i(y_1\varepsilon_1 + y_2\Lambda_0) \,\middle|\, \begin{matrix} x_1 = \frac{k}{2}x_2 \\ y_1 = \frac{k}{2}y_2 \end{matrix} \right\}$$



real part



complex part

Action of W is by reflections in hyperplanes

Shaded part is Y/W

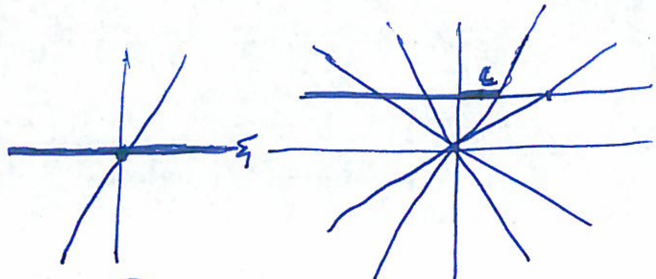
$T_0 =$   $T_1 =$  

real complex real complex.

Retract onto

$Y \to Y_i$

$z_1\xi_1 + z_2\Lambda_0 \longmapsto \frac{z_1}{z_2}\xi_1 + i\Lambda_0$.

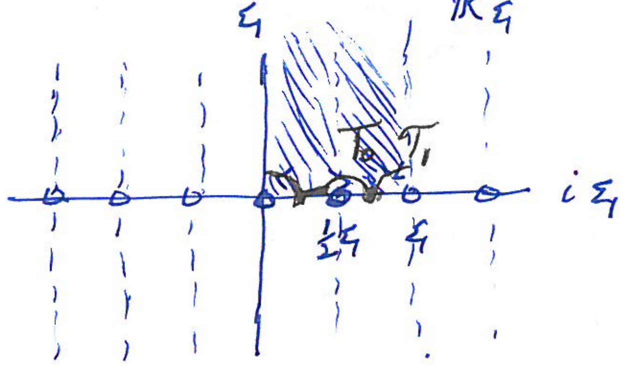


$\mathbb{R}\xi_1$

$i([0,1)\xi_1 + \Lambda_0)$.

$= i((\mathbb{R}-\mathbb{Z})\xi_1 + \Lambda_0)$.

So $Y_i =$

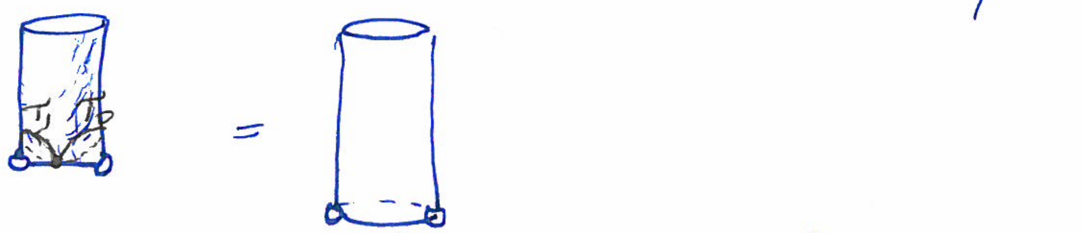


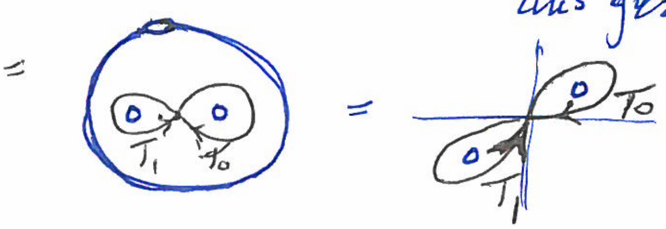
Y_i has W action where (a) translations do nothing on ξ_1-axis and $s_1\xi_1 = -\xi_1$

(b) translations translate on $i\xi_1$ axis and $s_1(i\xi_1) = -i\xi_1$

Translations have orbit the strips.

s_1 orbit leaves upper half of the strip.





So $\pi_1(Y/W)$ is a free group on 2 generators.

T_0 and T_1

Example of Affine Artingroup

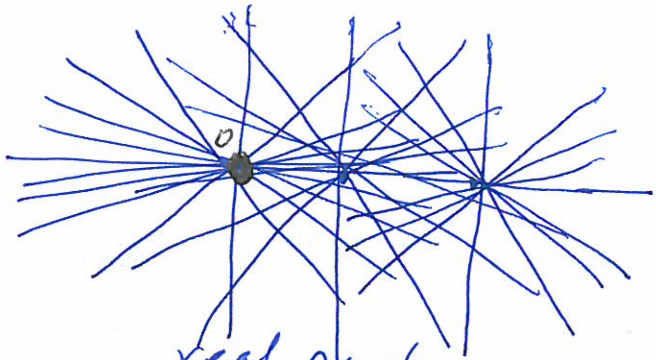
$N = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ So $W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$

$\alpha_1 = \varepsilon_1 \qquad \alpha_0 = -\varepsilon_1 + \frac{1}{2}\delta. \qquad Q = \mathbb{Z}\varepsilon_1 + \mathbb{Z}(-\varepsilon_1 + \frac{1}{2}\delta)$
$\qquad = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\frac{1}{2}\delta.$

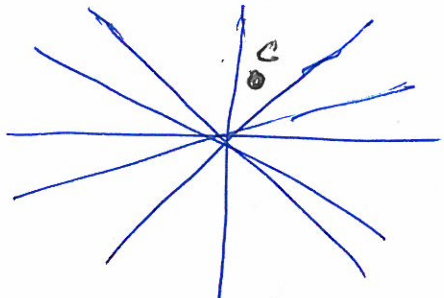
$\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\delta + \mathbb{R}\varepsilon_1 + \mathbb{R}\Lambda_0$ and $\mathcal{I}^\circ = \mathbb{R}\delta + \mathbb{R}\varepsilon_1 + \mathbb{R}_{>0}\Lambda_0$.

$$\mathcal{Y} = \mathfrak{h}_{\mathbb{R}}^* + i\mathcal{I}^\circ - \bigcup_{\substack{\alpha \in R^+ \\ m \in \mathbb{Z}}} \mathfrak{h}^{\alpha^\vee, m} + i\mathfrak{h}^{\alpha^\vee}$$

$$= \mathbb{R}\delta + \mathbb{R}\varepsilon_1 + \mathbb{R}\Lambda_0 + i(\mathbb{R}\delta + \mathbb{R}\varepsilon_1 + \mathbb{R}_{>0}\Lambda_0) - \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{Z}}} \mathfrak{h}^{\varepsilon_1 + \frac{k}{2}\delta, m} + i\mathfrak{h}^{\varepsilon_1 + \frac{k}{2}\delta}$$



real part

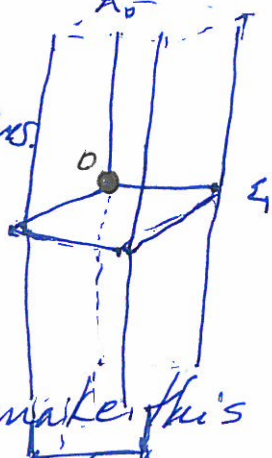


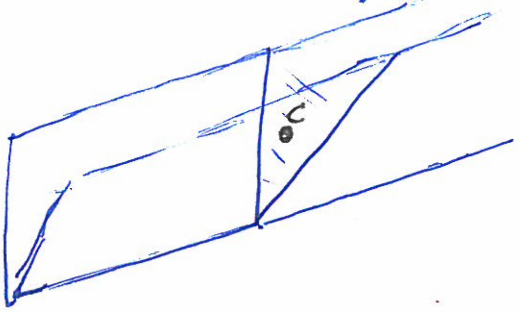
complex part

and the W-action has

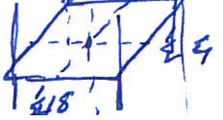
$t_{\varepsilon_1}(x+iy) = (\varepsilon_1 + x) + iy$ and $t_\delta(x+iy) = (\delta + x) + iy$.

Orbit representatives





Should we make this



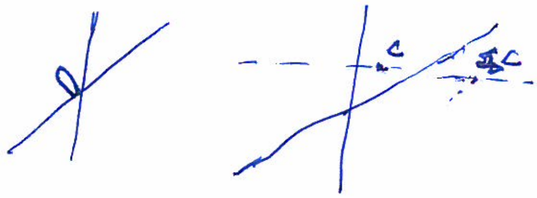
$$\zeta^{\xi_1+\frac{k}{2}\delta, m} + i\,\zeta^{\xi_1+\frac{k}{2}\delta}$$

$$= \left\{ \begin{array}{l} x_0\delta + x_1\xi_1 + x_2\Lambda_0 \\ + i(y_0\delta + y_1\xi_1 + y_2\Lambda_0) \end{array} \;\middle|\; \begin{array}{l} x_2\frac{k}{2} = x_1 + m \\ y_2\frac{k}{2} = y_1 \end{array} \right\}$$

$$= \left\{ z_0\delta + (m + \tfrac{k}{2}z_2)\xi_1 + z_2\Lambda_0 \;\middle|\; z_0, z_2 \in \mathbb{C} \right\}.$$

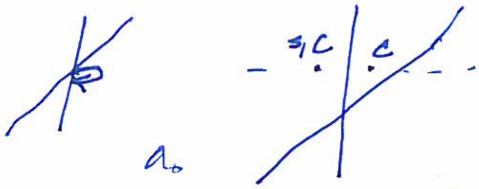
Now there are four paths

T_0:



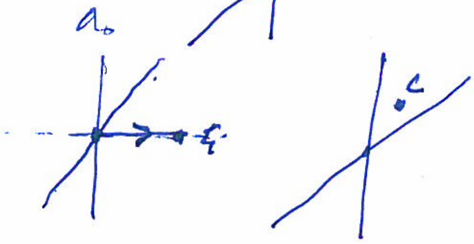
$s_0 c$ has some δ component.

T_1:

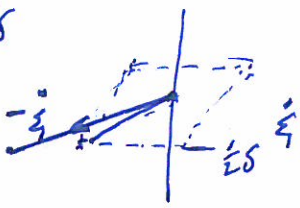


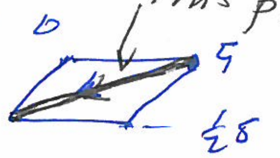
$s_1 c$ has 0 δ-component

$X^{\alpha_1} = X^{\xi_1}$

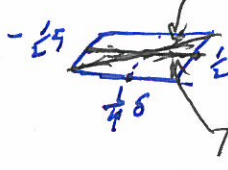


$X^{\alpha_0} = X^{-\xi_1+\frac{1}{2}\delta}$



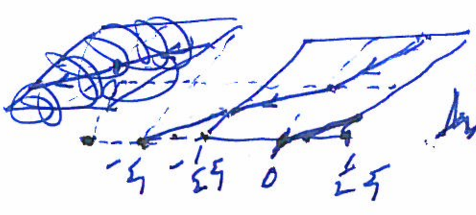


This path is X^{α_0}



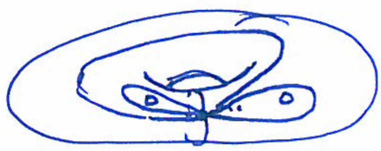
This is X^{α_0}

This is X^{α_1}



Why doesn't this end where it started?

We need a loop!!



$\mathcal{B} = \cancel{B_3} \times \mathbb{Z}$ has 4 generators

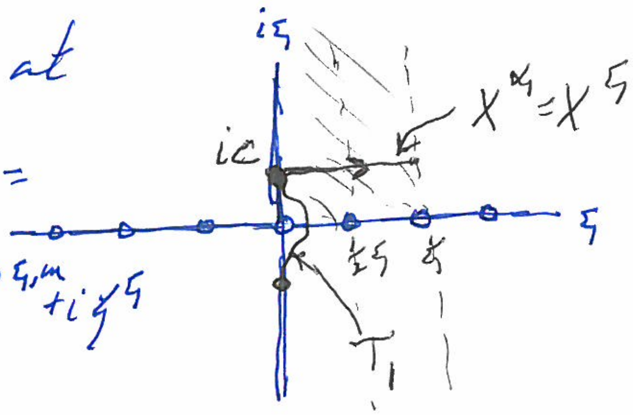
3 for the free group

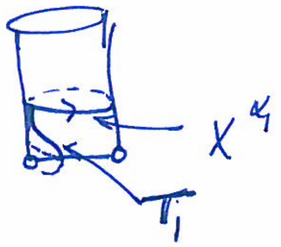
and one for q.

When we looked at

$\tilde{Y}_0 = \mathbb{C} - \mathbb{Z} =$

$= (\mathbb{R}_{\xi_1} + i\mathbb{R}_{\geq \xi_1}) - \bigcup_{m \in \mathbb{Z}} \xi^{\xi_1, m} + i\xi^{\xi}$





In general $\mathcal{B}$ has generators

$T_0, T_1, \dots, T_n$ and $X^{\alpha_0}, X^{\alpha_1}, \dots, X^{\alpha_n}$

i.e. $2(n+1)$ generators.