Lecture 1: Representation Theory, Reflection groups and Groups of Lie Type

Brazil Algebra Conference

Salvador 16 July 2012

Representation Theory

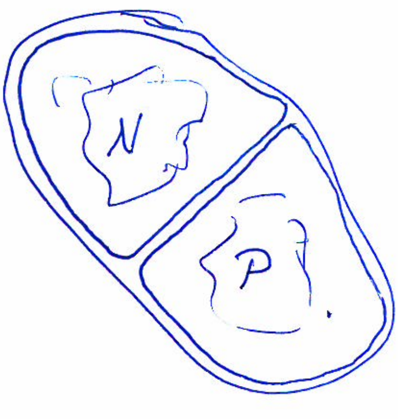
An <u>algebra</u> is a vector space with a product so that A is a ring.

<u>Representation theory</u> is the study of the category of A-modules (vector spaces M with an action of A).

A <u>simple A-module</u> is an A-module M with no submodules, except 0 and M.

<u>Problem</u>: Determine the simple A-modules.

An <u>indecomposable module</u> is an A-module M such that there <u>DOES NOT EXIST</u> N and P nonzero submodules with $M = N \oplus P$.



$M = N \oplus P$

$0 \to P \to M \to N \to 0$

but $M \neq N \oplus P$

Reflection groups Let Z be a subring of $\mathbb{C}$

A reflection group is a pair $(\mathfrak{h}_Z, W_0)$ with

$\mathfrak{h}_Z$ a free Z-module

W_0 a finite subgroup of $GL(\mathfrak{h}_Z)$
generated by reflections.

A reflection is an element $s \in GL_n(\mathbb{C})$ conjugate to

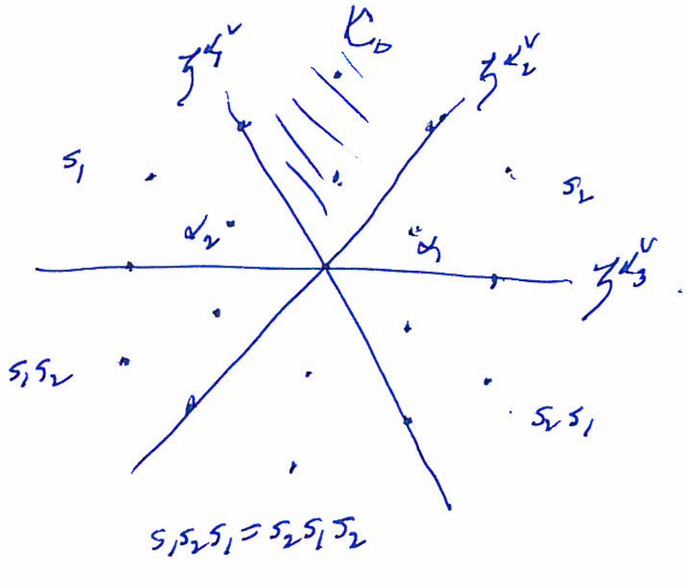
$$\begin{pmatrix} \xi & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad \text{with } \xi \neq 0.$$

A crystallographic reflection group is a $\mathbb{Z}$ reflection group.
A Euclidean reflection group is an $\mathbb{R}$-reflection group.

Examples

Type SL_3: $\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{span}\{\alpha_1, \alpha_2\}$

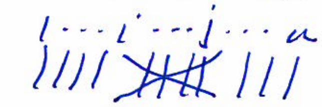
$W_0 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$



Type GL_n: $\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$.

$W_0 = S_n$ permuting $\varepsilon_1, \ldots, \varepsilon_n$

Reflections in S_n:

$s_{ij} =$  for $1 \le i < j \le n$.

$C_0 = \left\{ \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \;\middle|\; \begin{array}{l} \lambda_i \in \mathbb{R}, \\ \lambda_1 \ge \cdots \ge \lambda_n \end{array} \right\}$

<u>Coxeter's theorem</u> Let $(\mathfrak{h}_{\mathbb{R}}, W_0)$ be a Euclidean reflection group. Let C_0 be a fundamental region for W_0 acting on $\mathfrak{h}_{\mathbb{R}}$. Let

$\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ be the walls of C_0

$s_1, \ldots, s_n$ the corresponding reflections

Then W_0 is presented by generators $s_1, \ldots, s_n$ with relations

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} \quad \text{for } i \neq j$$

where $\pi/m_{ij} = \mathfrak{h}^{\alpha_i^\vee} \measuredangle \mathfrak{h}^{\alpha_j^\vee}$.

<u>Groups of Lie Type</u>

<u>Theorem</u> <u>Type GL_n</u> $GL_n(\mathbb{C})$ is generated by

$$x_{ij}(c) = \begin{pmatrix} 1 & & c \\ & \ddots & \\ & & 1 \end{pmatrix} \qquad x_{ji}(c) = \begin{array}{c c} & i \\ j & \begin{pmatrix} 1 & & \\ c & \ddots & \\ & & 1 \end{pmatrix} \end{array}$$

$$h_\lambda(t) = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix} \quad \text{for} \quad \begin{array}{l} 1 \le i < j \le n, \quad c \in \mathbb{C} \\ \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \in \mathfrak{h}_{\mathbb{Z}}, \quad t \in \mathbb{C}^\times \end{array}$$

with relations

$$x_{ij}(c_1) x_{ij}(c_2) = x_{ij}(c_1 + c_2), \ \ldots$$

$$w x_{ij}(c) w^{-1} = x_{w(i)w(j)}(c), \qquad w h_\lambda(t) w^{-1} = h_{w\lambda}(t)$$

and more ....

<u>Theorem</u> <u>Type SL_3</u> $SL_3(\mathbb{F})$ is generated by

$$x_{\alpha_1^\vee}(c) = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix} \quad x_{\alpha_2^\vee}(c) = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix} \quad x_{\alpha_3^\vee}(c) = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$x_{-\alpha_1^\vee}(c) = \begin{pmatrix} 1 & & \\ c & 1 & \\ & & 1 \end{pmatrix} \quad x_{-\alpha_2^\vee}(c) = \begin{pmatrix} 1 & & \\ & 1 & \\ & c & 1 \end{pmatrix} \quad x_{-\alpha_3^\vee}(c) = \begin{pmatrix} 1 & & \\ & 1 & \\ c & & 1 \end{pmatrix}$$

$$h_{\alpha_1}(t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix} \quad h_{\alpha_2}(c) = \begin{pmatrix} 1 & & \\ & t & \\ & & t^{-1} \end{pmatrix}$$

with relations

$$x_{\alpha_1^\vee}(c_1)x_{\alpha_1^\vee}(c_2) = x_{\alpha_1^\vee}(c_1+c_2), \quad h_{\alpha_1}(t_1)h_{\alpha_1}(t_2) = h_{\alpha_1}(t_1t_2)$$

etc.

<u>Theorem</u> (Chevalley-Steinberg-Tits) Let $(\mathfrak{h}_\mathbb{Z}, W_0)$ be a crystallographic reflection group.

R^+ an index set for the reflections in W_0

Define G by generators

$x_\alpha(c)$, $x_{-\alpha}(c)$ and $h_\lambda(t)$, for $\alpha \in R^+$, $c \in \mathbb{C}$, $\lambda \in \mathfrak{h}_\mathbb{Z}$, $t \in \mathbb{C}^\times$

with relations

$$x_\alpha(c_1)x_\alpha(c_2) = x_\alpha(c_1+c_2),$$

$$h_\lambda(t_1)h_\lambda(t_2) = h_\lambda(t_1t_2), \quad h_\lambda(t)h_\mu(t) = h_{\lambda+\mu}(t),$$

and more.

~~Theore~~

$$\left\{\begin{matrix}\mathbb{Z}\text{-reflection groups} \\ (\mathfrak{t}_{\mathbb{Z}}, W_0)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{complex reductive} \\ \text{algebraic groups}\end{matrix}\right\}$$

is an equivalence of categories.

<u>Other equivalences</u>

$$\left\{\begin{matrix}\text{complex reductive} \\ \text{algebraic groups}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{compact Lie} \\ \text{groups}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{connected, simply} \\ \text{connected, compact} \\ \text{Lie groups}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{complex semisimple} \\ \text{Lie algebras}\end{matrix}\right\}$$

Andersen-Grodal et al have proved:

There is an equivalence of categories:

$$\{\mathbb{Z}_p\text{-reflection groups}\} \longrightarrow \{p\text{-compact groups}\}$$

$$(\mathfrak{t}_{\mathbb{Z}_p}, W_0) \longmapsto BG$$