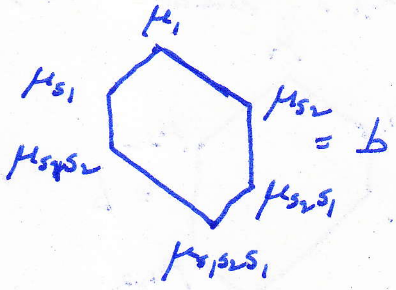
In progress with A. Ghitza and S. Kannan

Compare $H^0(G/B, \mathcal{L}_\lambda)$ and MV cycles of type λ.

column strict tableaux

1	1	1	2	2
2	3	3	4	5
3	4	5		

$\overset{1-1}{\longleftrightarrow}$



MV polytopes

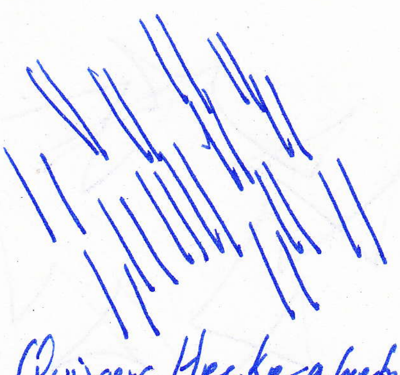
$\updownarrow$

elements of the shuffle algebra $\mathbb{C}[N]$

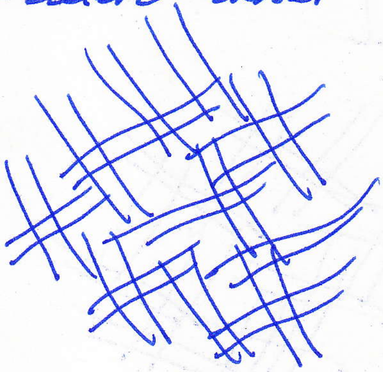
$ch(L_b)$	$ch(\Lambda_b)$	$ch(Z_b)$
dual canonical basis	dual semicanonical basis	MV basis

Lusztig, Khovanov-Lauda-Rouquier, Leclerc

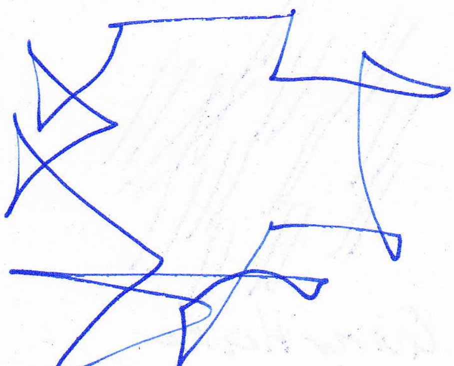
Lusztig, Kashiwara-Saito, Geiss-Leclerc-Schröer



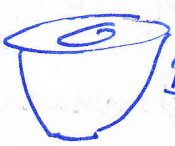
Quiver Hecke algebra modules L_b



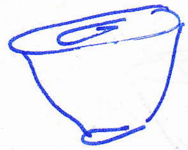
Preprojective algebra modules Λ_b



MV cycles Z_b



Baumann-Kamnitzer



Kamnitzer

The shuffle algebra $\mathbb{C}[N]$

Let $\mathcal{F}$ be the free algebra generated by $f_1, \ldots, f_n$

The shuffle product $\circ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is given by

$$u \circ v = \sum_{\sigma \in S_{k+l}/S_k \times S_l} \sigma(uv), \quad \text{for words} \quad \begin{array}{l} u = f_{i_1} \cdots f_{i_k} \\ v = f_{j_1} \cdots f_{j_l} \end{array}$$

where the sum is over minimal length coset representatives for cosets in $S_{k+l}/S_k \times S_l$. For example

$$f_1f_2 \circ f_2f_1 = \underline{f_1}\,\underline{f_2}f_2f_1 + \underline{f_1}f_2\underline{f_2}f_1 + \underline{f_1}f_2f_1\underline{f_2} + f_2\underline{f_1}\,\underline{f_2}f_1 + f_2\underline{f_1}f_1\underline{f_2} + f_2f_1\underline{f_1}\,\underline{f_2}$$

$\mathbb{C}[N]$ is the $\circ$-subalgebra of $\mathcal{F}$ generated by $f_1, \ldots, f_n$.

$f_1, \ldots, f_n$ correspond to simple coroots $\alpha_1^\vee, \ldots, \alpha_n^\vee$ for a root system.

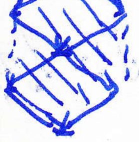
Favorite root system: $\alpha_i = \alpha_i^\vee = \varepsilon_i - \varepsilon_{i+1}$, $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$.

$$W_0 = S_n, \qquad s_i = \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}, \qquad w_0 = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}$$

The MV polytope corresponding to $\mathrm{ch}(L_b)$ is the convex hull of the paths/terms in $\mathrm{ch}(L_b)$.

Example: 

and $\mathrm{ch}(L_b) = f_1f_2 \circ f_2f_2$ has MV-polytope

$b =$ 

Quiver Hecke algebra modules

The Khovanov-Lauda-Rouquier, or quiver Hecke, algebra R_d has generators

$$e_u, \quad y_1, \ldots, y_d, \quad \psi_1, \ldots, \psi_{d-1}$$

where $u = f_{i_1} \cdots f_{i_d}$ runs over words of length d

$$e_u e_v = \delta_{uv} e_u \quad \text{and} \quad \sum_u e_u = 1$$

$y_1, \ldots, y_d$ are like Murphy elements

$\psi_1, \ldots, \psi_{d-1}$ are like simple transpositions in S_d.

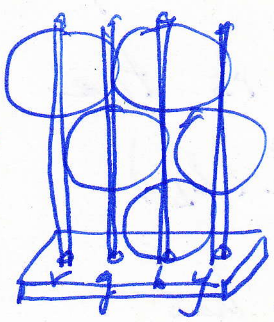
R_d is $\mathbb{Z}$-graded! For a $\mathbb{Z}$-graded R_d-module M,

$$M = \bigoplus_i M[i] = \bigoplus_i \bigoplus_u e_u M[i]$$

and the character of M is

$$ch(M) = \sum_i \sum_u \dim(e_u M[i])\, q^i u$$ in the q-shuffle algebra.

Kleshchev-Ram: The simple homogeneous R_5 module corresponding to

$b =$ 

has dimension the number of standard tableaux of shape b

5 4 / 2 3 / 1   4 5 / 2 3 / 1   5 4 / 3 2 / 1   4 5 / 3 2 / 1   3 5 / 2 4 / 1

and

$$ch(L_b) = f_b f_g f_y f_b f_r + f_b f_g f_y f_r f_b + f_b f_y f_g f_b f_r + f_b f_y f_g f_r f_b + f_b f_g f_r f_y f_b.$$

MV polytopes

An <u>MV polytope</u> is

$$b = \text{convex hull}\{\mu_w \mid w \in W_0\}, \text{ its vertices.}$$

For a minimal length path $w_0 = s_{i_1} \cdots s_{i_N}$ to w_0 the <u>i-perimeter</u> or <u>Lusztig parametrization</u> of b is

$$\mathrm{per}_{\underline{i}}(b) = (l_1, l_2, \ldots, l_N),$$

the sequence of lengths $\mu_1 \xrightarrow{l_1} \mu_{s_{i_1}} \xrightarrow{l_2} \mu_{s_{i_1}s_{i_2}} \longrightarrow \cdots$

Any $\mathrm{per}_{\underline{j}}(b)$ can be computed from $\mathrm{per}_{\underline{i}}(b)$ by a sequence of "Coxeter relations":

$$R_{i\,i+1\,i}^{i+1\,i\,i+1}(l_a, l_{a+1}, l_{a+2}) = (l_{a+1} + l_{a+2} - \min(l_a, l_{a+2}), \min(l_a, l_{a+2}), l_a + l_{a+1} - \min(l_a, l_{a+2}))$$

$$R_{ij}^{ji}(l_a, l_{a+1}) = (l_{a+1}, l_a)$$

(see Morier-Genoud Theses).

The <u>crystal operator</u> $\tilde{f}_{i_1}$ is given by

$$\mathrm{per}_{\underline{i}}(\tilde{f}_{i_1} b) = (l_1 + 1, l_2, \ldots, l_N)$$

and the <u>i-growth</u>, or <u>string parametrization</u> of b is

$$b = \tilde{f}_{i_1}^{c_1} \cdots \tilde{f}_{i_N}^{c_N} b_+, \quad \text{where } b_+ = \bullet.$$

~~The~~ MV cycles

$$\mathbb{C}((t)) = \{a_{-c}t^{-c} + a_{-c+1}t^{-c+1} + \cdots \mid a_i \in \mathbb{C},\ -c \in \mathbb{Z}\}$$
$$\cup\!\!\mid$$
$$\mathbb{C}[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C}\}$$

$$G = GL_{n+1}(\mathbb{C}((t))) \qquad K = GL_{n+1}(\mathbb{C}[[t]])$$

$$U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\} \subseteq GL_{n+1}(\mathbb{C}((t))).$$

Let

$$t_{\lambda^\vee} = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_n} \end{pmatrix} \qquad y_i(at^j) = \begin{array}{c} \\ \\ i \\ i+1 \\ \\ \end{array} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & at^j & 1 & \\ 0 & & & & \ddots \ 1 \end{pmatrix}$$

for $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ and $a \in \mathbb{C}$, $j \in \mathbb{Z}$.

G/K is the loop Grassmannian.

The Cartan and Iwasawa decompositions are

$$G = \bigsqcup_{\lambda^\vee} K t_{\lambda^\vee} K \quad \text{and} \quad G = \bigsqcup_{\mu^\vee} U^- t_{\mu^\vee} K$$

The MV cycles of type $\lambda^\vee$ and weight $\mu^\vee$ are the irreducible components

$$Z_b \in \mathrm{Irr}\left(\overline{K t_{\lambda^\vee} K \cap U^- t_{\mu^\vee} K} \right).$$

Composition series: $ch(Z_b)$

The MV cycles are indexed by MV polytopes and by Baumann-Gaussent,

$$\text{if} \quad b = \tilde{f}_{i_1}^{c_1} \cdots \tilde{f}_{i_N}^{c_N} b_+ \quad \text{then}$$

$$Z_b = \overline{y_{i_1}(t^{e_1}\mathbb{C}[t^{-1}]^\times_{c_N}) \cdots y_{i_d}(t^{e_N}\mathbb{C}[t^{-1}]^\times_{c_N})K},$$

where

$$e_j = \langle \alpha_{i_j}, -c_{j+1}\alpha_{i_{j+1}} - \cdots - c_N \alpha_{i_N} \rangle \quad \text{and}$$

$$\mathbb{C}[t^{-1}]^\times_c = \{ a_{-c}t^{-c} + \cdots + a_{-2}t^{-2} + a_{-1}t^{-1} \mid a_i \in \mathbb{C},\ a_{-c} \in \mathbb{C}^\times \}$$

Let Z_b be an MV cycle of dimension d.

A <u>composition series</u> for Z_b is

$$(i_1, \ldots, i_d \mid j_1, \ldots, j_d)$$

such that

$$Z_b = \overline{\{ y_{i_1}(a_1 t^{j_1}) \cdots y_{i_d}(a_d t^{j_d})K \mid a_i \in \mathbb{C} \}}$$

The <u>character</u> of Z_b is

$$ch(Z_b) = \sum_{(i_1,\ldots,i_d \mid j_1,\ldots,j_d)} f_{i_1} \cdots f_{i_d} \quad \text{an element of } \mathbb{C}[N].$$