On the cohomology and K-theory of G/B

Working seminar 15. 12. 2010 Univ. of Melbourne

The flag variety G/B

Reading: Lam, Schilling, Shimozono, arXiv 0901.1506.

G a symmetrizable Kac-Moody group

$\cup|$

B a Borel subgroup

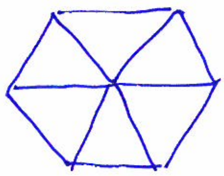
or some appropriate variant

Then
$$G = \bigsqcup_{w \in W} BwB, \quad \text{where } W \text{ is the Weyl group.}$$

Examples

(1) $G = SL_3(\mathbb{C})$ and $B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subseteq SL_3(\mathbb{C})$

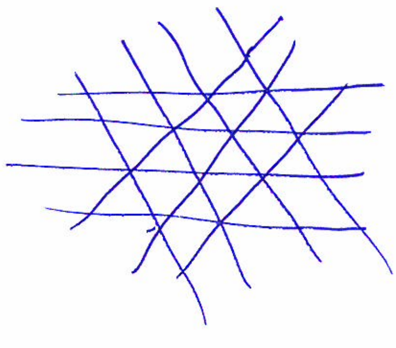
Then W is the set of chambers



(2) $G = SL_3(\mathbb{C}((t)))$ and

$$I = \{ g \in SL_3(\mathbb{C}[[t]]) \mid g|_{t=0} \in B \}$$

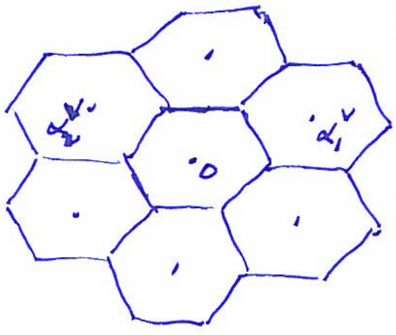
Then W is the set of chambers



(3) $G = SL_3(\mathbb{C}((t)))$ and

$$K = SL_3(\mathbb{C}[[t]]).$$

Then W is the set of hexagons



$Q^\vee = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \alpha_2^\vee\}$

and $P = \{ \lambda : Q^\vee \to \mathbb{Z} \mid \lambda \text{ is } \mathbb{Z}\text{-linear} \}$

The Nil-Hecke ring

Let
$$R(T) = \mathrm{span}\{e^\lambda \mid \lambda \in P\} \quad \text{with} \quad e^\lambda e^\mu = e^{\lambda+\mu}$$
$$\mathbb{C}W_0 = \mathrm{span}\{t_w \mid w \in W_0\} \quad \text{with} \quad t_u t_v = t_{uv}$$
and
$$K = \mathbb{C}(W_0 \ltimes P) = \mathrm{span}\{e^\lambda t_w \mid \lambda \in P, w \in W\} \quad \text{with}$$
$$t_w e^\lambda = e^{w\lambda} t_w.$$

K acts on $R(T)$ by
$$e^\lambda e^\mu = e^{\lambda+\mu} \quad \text{and} \quad t_w e^\lambda = e^{w\lambda}.$$
The coproduct on K is $\Delta : K \otimes_{R(T)} K \longrightarrow R(T)$ given by
$$\Delta(e^\lambda) = e^\lambda \otimes 1$$
$$\Delta(t_w) = t_w \otimes t_w$$
and $pm \otimes n = m \otimes pn$ for $p \in R(T)$, $m, n \in K$.

Let
$$y_i = \frac{1}{1-e^{-\alpha_i}}(1 - e^{-\alpha_i} s_i) \quad \text{and} \quad T_i = \frac{1}{1-e^{\alpha_i}}(s_i - 1)$$
Then
$$y_i^2 = y_i \qquad\qquad T_i^2 = -T_i$$
$$y_i y_j y_i \cdots = y_j y_i y_j \cdots \qquad\qquad T_i T_j T_i \cdots = T_j T_i T_j \cdots$$
and
$$\Delta(T_i) = 1 \otimes T_i + T_i \otimes 1 + (1 - e^{\alpha_i})(T_i \otimes T_i).$$

K-cohomology $K^T(G/B)$

In K write

$$w = \sum_{v \in W} \psi^v(w) T_w \qquad \text{so that } \psi^v \in Fun(W, R(T))$$

Then $Fun(W, R(T))$ is the dual of K by the pairing

$$\langle \psi^v, T_w \rangle = \delta_{v,w},$$

so that $Fun(W, R(T)) = R(T)\text{-span}\{\psi^v \mid v \in W\}$.

The GKM condition

$$\Psi = \left\{ \psi \in Fun(W, Q(T)) \;\middle|\; \begin{array}{l} \psi(s_\alpha w) - \psi(w) \in (1-e^{\alpha})R(T) \\ \text{for } \alpha \in R^+_{re} \text{ and } w \in W \end{array} \right\}$$

$$= \{\psi \in Fun(W, Q(T)) \mid \psi(T_w) \in R(T) \text{ for } w \in W\}$$

$K^T(G/B)$ has $K^T(pt)$-basis $\{[\mathcal{O}_{X_w}] \mid w \in W\}$

Let $\iota_w : pt \to G/B$ be the T-fixed pt wB in G/B and define

$$\psi^v(w) = \iota_w^*([\mathcal{O}_{X_v}]).$$

Then

$$\begin{array}{ccc} K^T(G/B) & \longrightarrow & \Psi \\ [\mathcal{O}_{X_v}] & \longmapsto & \psi^v \end{array} \quad \text{is an isomorphism.}$$

Demazure operators

$$p_i : G/B \longrightarrow G/P_i \quad \text{where} \quad W_i = \langle s_i \rangle.$$

Then

$$\begin{tikzcd} K^T(G/B) \arrow[r, "\mathrm{res}"] \arrow[d, "p_i^* p_{i*}"'] & \Psi \arrow[d, "y_i \cdot"] \\ K^T(G/B) \arrow[r, "\mathrm{res}"'] & \Psi \end{tikzcd}$$

where $(y_i \cdot \psi)(w) = \psi(w y_i)$
and $p\psi(w) = \psi(pw)$
for $p \in R(T)$.

Partial flag varieties

Let $J \subseteq \{s_1, \ldots, s_n\}$ and $W_J = \langle s_j \mid j \in J \rangle$.

P_J is the parabolic corresponding to W_J

W^J is the set of (minimal length) coset representative of W/W_J

Define $\iota_J : \mathrm{Fun}(W^J, Q(T)) \longrightarrow \mathrm{Fun}(W, Q(T))$

by

$$\iota_J(\psi_J^v) = \psi^v \quad \text{for} \quad v \in W^J.$$

If $p_J : G/B \longrightarrow G/P_J$ then

$$\begin{tikzcd} K^T(G/P_J) \arrow[r, "\mathrm{res}"] \arrow[d, "p_J^*"'] & \Psi^J \arrow[d, "\iota_J"] \\ K^T(G/B) \arrow[r, "\mathrm{res}."'] & \Psi \end{tikzcd}$$

The Peterson subalgebra

Let $\mathcal{L} = \bigoplus_{\lambda \in Q^\vee} \mathbb{Q}(T) t_\lambda \cap K$, a Hopf subalgebra of K.

Let

$\varpi : \Psi_{af} \longrightarrow \Psi_{af}^{I}$ be given by

$(\varpi \Psi)(w) = \Psi(t_\lambda)$, where $wW = t_\lambda W$.

(elements of Ψ_{af}^{I} are functions on W constant on hexagons
– this is the natural embedding $\Psi_{af}^{I} \hookrightarrow \Psi_{af}$)

ϖ is an $R(T)$ module homomorphism.

Define $k : K_T(Gr_G) \longrightarrow R(T)\text{-span}\{T_w \mid w \in W_{af}\} = K$.

by $\langle k(\xi), \Psi \rangle = \langle \xi, \varpi(\Psi) \rangle$. (identifying $\Psi_{af}^{I} \cong K^T(Gr_G)$)

Then

$k : K_T(Gr_G) \xrightarrow{\sim} \mathcal{L}$ as Hopf algebras.

Remark $\mathcal{L} = \operatorname{im} k$

$= Z_K(R(T)) =$ centralizer of $R(T)$ in K