

What are KLR BMW algebras? International Conference on
Non-commutative rings and combinatorial
The affine braid group B_k Representation Theory ①

$$\delta = \begin{array}{c} \text{Diagram of } \delta \\ \text{in } B_4 \end{array} \quad \text{and} \quad \delta_1, \delta_2 = \begin{array}{c} \text{Diagram of } \delta_1 \\ \text{in } B_4 \\ \text{and} \\ \text{Diagram of } \delta_2 \\ \text{in } B_4 \end{array}$$

Fix constants q and z . Let

$$T_i = \begin{array}{c} \text{Diagram of } T_i \\ \text{in } B_4 \end{array} \quad \text{and} \quad Y_i = z \begin{array}{c} \text{Diagram of } Y_i \\ \text{in } B_4 \end{array}$$

$$E_i = \begin{array}{c} \text{Diagram of } E_i \\ \text{in } B_4 \end{array} \quad \text{given by} \quad T_i Y_i = Y_{i+1} T_i - (q - q^{-1}) (1 - E_i).$$

The affine BMW algebra \mathcal{W}_k is $C B_k$ with

$$\varphi = z^{-1}, \quad \psi = z, \quad \begin{array}{c} \text{Diagram of } \varphi \\ \text{in } B_n \end{array} = \prod_n, \quad \begin{array}{c} \text{Diagram of } \psi \\ \text{in } B_n \end{array} = \prod^n$$

and

$$\text{twists} \quad \left\{ \begin{array}{c} \text{Diagram of twists} \\ \text{in } B_n \end{array} \right\} = z_i^{(k)} \quad \text{where } z_i^{(k)} \text{ are constants}$$

The affine Hecke algebra \mathcal{H}_k is \mathcal{W}_k with

$$E_i = 0.$$

For some choices of $z_i^{(k)}$, $\mathcal{W}_k = \mathcal{H}_k$.

(2)

The degenerate affine braid group B_k is the algebra generated by

$$t_{s_i} = \begin{array}{c} / / / / \\ \times \end{array} \quad \text{and} \quad y_i = \begin{array}{c} / / / / \\ \phi \end{array}$$

with relations

$$\begin{array}{c} X \\ X \end{array} \cdot \begin{array}{c} X \\ X \end{array} = \begin{array}{c} X \\ X \end{array}, \quad \begin{array}{c} X \\ X \end{array} \cdot \begin{array}{c} X \\ X \end{array} = \begin{array}{c} X \\ X \end{array}, \quad \begin{array}{c} X \\ X \end{array} = \begin{array}{c} / / \\ / / \end{array}$$

$$y_i \cdot y_j = y_j \cdot y_i \quad \text{and} \quad t_{s_i} \cdot (y_i + y_{i+1}) = (y_i + y_{i+1}) t_{s_i}$$

if $t_{i,i+1} = y_{i+1} - t_{s_i} y_i t_{s_i}$ then $t_{s_i} t_{s_{i+1}} (t_{i,i+1}) t_{s_{i+1}} t_{s_i} = t_{i+1, i+2}$

Fix $\epsilon = \pm 1$ and define $e_i = \begin{array}{c} / / / / \\ \epsilon \end{array}$ by

$$t_{s_i} y_i = y_{i+1} t_{s_i} - (1 - e_i)$$

The degenerate affine BMW algebra is B_k with

$$t_{s_i} e_i = \epsilon e_i = e_i t_{s_i}, \quad e_i \cdot t_{s_{i+1}} e_i = \epsilon e_i.$$

$$e_i \cdot (y_i + y_{i+1}) = (y_i + y_{i+1}) e_i = 0 \quad \text{and}$$

$$e_i y_i^l e_i = z_i^{(l)} e_i \quad \text{where } z_i^{(l)} \text{ are constants.}$$

The degenerate affine Hecke algebra H_k is W_k

with $e_i = 0$.

KLR algebras

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Let $I = \mathbb{Z}/c\mathbb{Z}$, $c \in \mathbb{Z}_{\geq 0}, c \neq 2$ ($c=0$ is the most. imp. case).

Fix symbols α_i , $i \in I$, and

$$\alpha = \sum_{i \in I} m_i x_i, \text{ with } m_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{i \in I} m_i = k.$$

The KLR algebra R_α has generators

$$P_{i_1 \dots i_k} = \frac{111111}{i_1 i_2 \dots i_k}, \quad x_i = 11110111, \quad \psi = 1111X111$$

for $x_1 + \dots + x_k = x$, with relations

$$P_U P_V = \delta_{UV} P_U \quad \sum_{U=i_1 \dots i_k} P_U = 1, \quad x_i x_j = x_j x_i, \quad x_i P_U = P_U x_i$$

$x_{i_1} + \dots + x_{i_k} = \alpha$

$$\psi_L \rho_\mu = \rho_{\sigma L} \psi_L, \quad \text{i.e.} \quad X_{ab} = {}^b X^a$$

$$\begin{aligned} X_{aa} &= X_{aa} + II, \quad X_{aa} = X_{aa} - II, \quad X_{ab} = X_{ab}, \quad X_{ab} = X_{ab} \\ X_{bb} &= X_{bb} + II, \quad X_{bb} = X_{bb} - II, \quad X_{ba} = X_{ba}, \quad X_{ba} = X_{ba} \end{aligned}$$

$$\cancel{a+a+a} = \cancel{a+a} + \cancel{a}, \quad \cancel{a+a+a} = \cancel{a+a} - \cancel{a}, \quad \cancel{ab}c = \cancel{ab}c$$

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MOST IMPORTANT

(1) R_α is a \mathbb{Z} -graded algebra

$$\deg \left(\prod_{i_1, i_2, \dots, i_k} \right) = 0, \quad \deg \left(\prod_{i_1, i_2, \dots, i_k} \right) = 2$$

$$\deg(X_{aa}) = -2, \quad \deg(X_{a^{at+1}}) = 1, \quad \deg(X_{ab}) = 0.$$

(2) There is a character map.

Let F be the free associative algebra generated by $f_i, i \in I$.

Let M be a \mathbb{Z} -graded R_α -module $M = \bigoplus_{l \in \mathbb{Z}} M[l]$

$$\text{char}(M) = \sum_{l \in \mathbb{Z}} \sum_{\substack{i_1, \dots, i_k \\ i_1 + \dots + i_k = l}} \dim(\varphi_{i_1, \dots, i_k} M[l]) q^l f_{i_1} \cdots f_{i_k}$$

so that $\text{char}: \begin{cases} \text{\mathbb{Z}-graded} \\ \text{R_α-modules} \end{cases} \rightarrow F$

The multiplication on $\{\text{$\mathbb{Z}$-graded R_α-modules}\}$ is

$$M \circ N = \text{Ind}_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M \otimes N)$$

and the induced product on F is the q -shuffle product.
 The image of char is the quantum group.

(5)

What are KLR BMW algebras?

P_α is generated by

$$\varphi_{i_1 \dots i_k} = \underset{i_1 i_2 \dots i_k}{\text{|||||}}, \quad x_i = \text{|||} \uparrow \text{|||}, \quad \psi_L = \text{|||} X \text{|||}, \quad \kappa_L = \text{|||} \swarrow \text{|||}$$

(2) is obtained by the key lemma

$R_{\alpha\beta}$ is a free $R_\alpha \otimes R_\beta$ module with basis

$$\{\psi_w \mid w \in S_{k+l}/S_k \times S_l\} \text{ where}$$

$$\psi_w = \psi_{l_1} \dots \psi_{l_r} \text{ for a fixed reduced word } w = s_{l_1} \dots s_{l_r}$$

We need

$$(1) \deg \left(\begin{smallmatrix} c & d \\ \uparrow & \downarrow \\ a & b \end{smallmatrix} \right) = ???$$

and relations

$$a^\alpha b^\beta = 0 \text{ unless } \beta = r - \alpha,$$

$$\cap \stackrel{?}{=} - \cap, \quad \times = ??? \cap, \quad \times \square = ??? \mid$$

$$\left[\begin{array}{c} \vdots \\ \vdots \end{array} \right] = ???$$

We want

$$\bigoplus_{\substack{\alpha \\ h(\alpha)=k}} P_\alpha \stackrel{?}{=} W_k \text{ where}$$

" $\stackrel{?}{=}$ " means 'have the same finite dimensional representations'.

$$\text{where } h(\alpha)=k \text{ with } \alpha = \sum_{i \in I} m_i x_i \text{ with } \sum_{i \in I} m_i = k.$$