A combinatorial formula for Macdonald polynomials

(joint work with Martha Yip)

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Equivalences

{connected compact Lie groups}	$SU(n)$
$\updownarrow$	
{connected complex reductive algebraic groups}	$SL_n(\mathbb{C})$
$\updownarrow$	
{complex semisimple Lie algebras}	$\mathfrak{sl}_n$
$\updownarrow$	
{root systems}	$\{\varepsilon_i - \varepsilon_j\} \subseteq \mathbb{R}^n$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \text{for } 1 \leq i \leq n.$

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{root systems}	$\{\varepsilon_i - \varepsilon_j\} \subseteq \mathbb{R}^n / \langle \varepsilon_1 + \cdots + \varepsilon_n \rangle$

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Equivalences

{connected compact Lie groups}	$U(n)$
$\updownarrow$	
{connected complex semisimple algebraic groups}	$GL_n(\mathbb{C})$
$\updownarrow$	
{complex reductive Lie algebras}	$\mathfrak{gl}_n$
$\updownarrow$	
{root systems}	$\{\varepsilon_i - \varepsilon_j\} \subseteq \mathbb{R}^n$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad \text{for } 1 \leq i \leq n.$

Chevalley says:

$$\{\text{connected compact Lie groups}\} \longleftrightarrow \{\mathbb{Z}\text{-reflection groups}\}$$

$$G \longmapsto (W_0, \mathfrak{h}^*_{\mathbb{Z}})$$

Anderson, Grodal, et al say:

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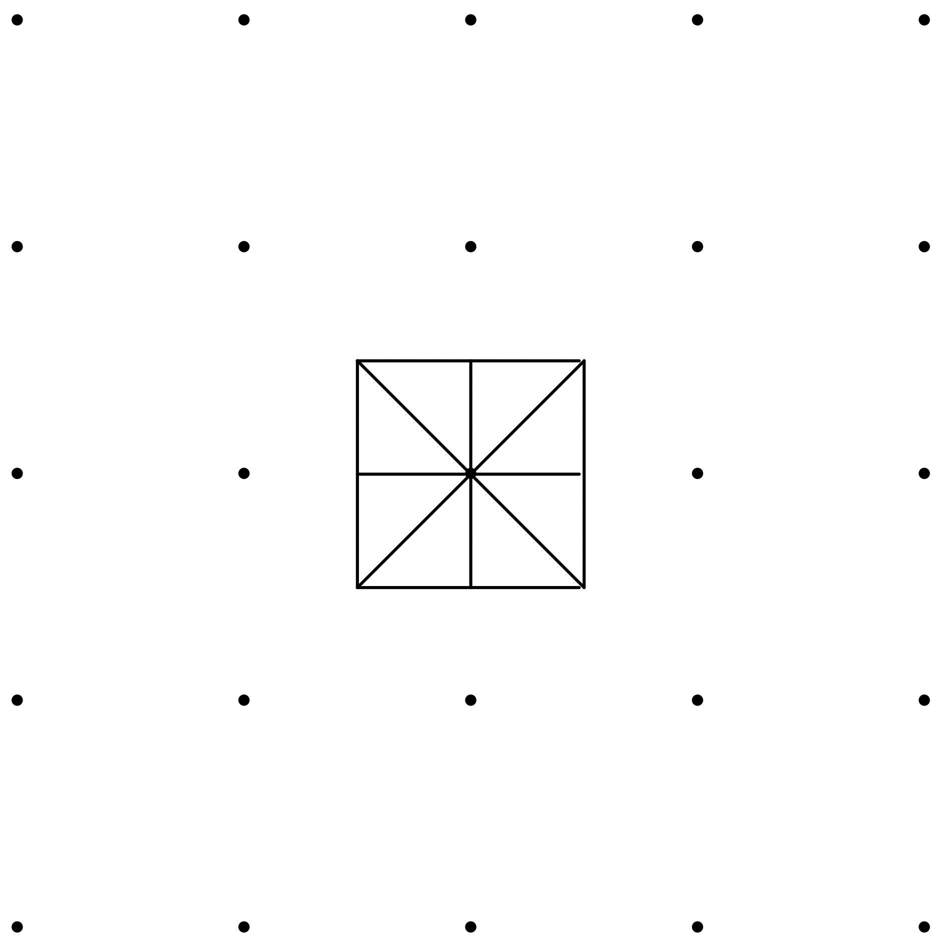
Compact Lie group $= (W_0, \mathfrak{h}^*_{\mathbb{Z}})$



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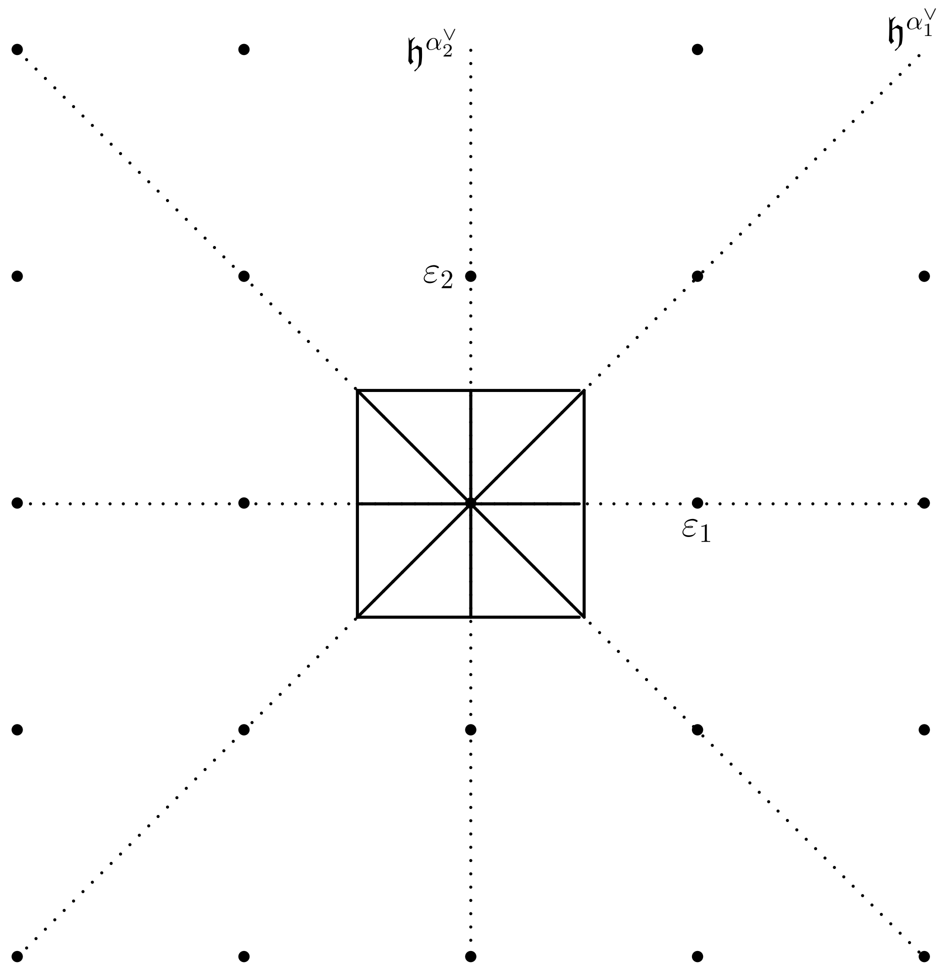

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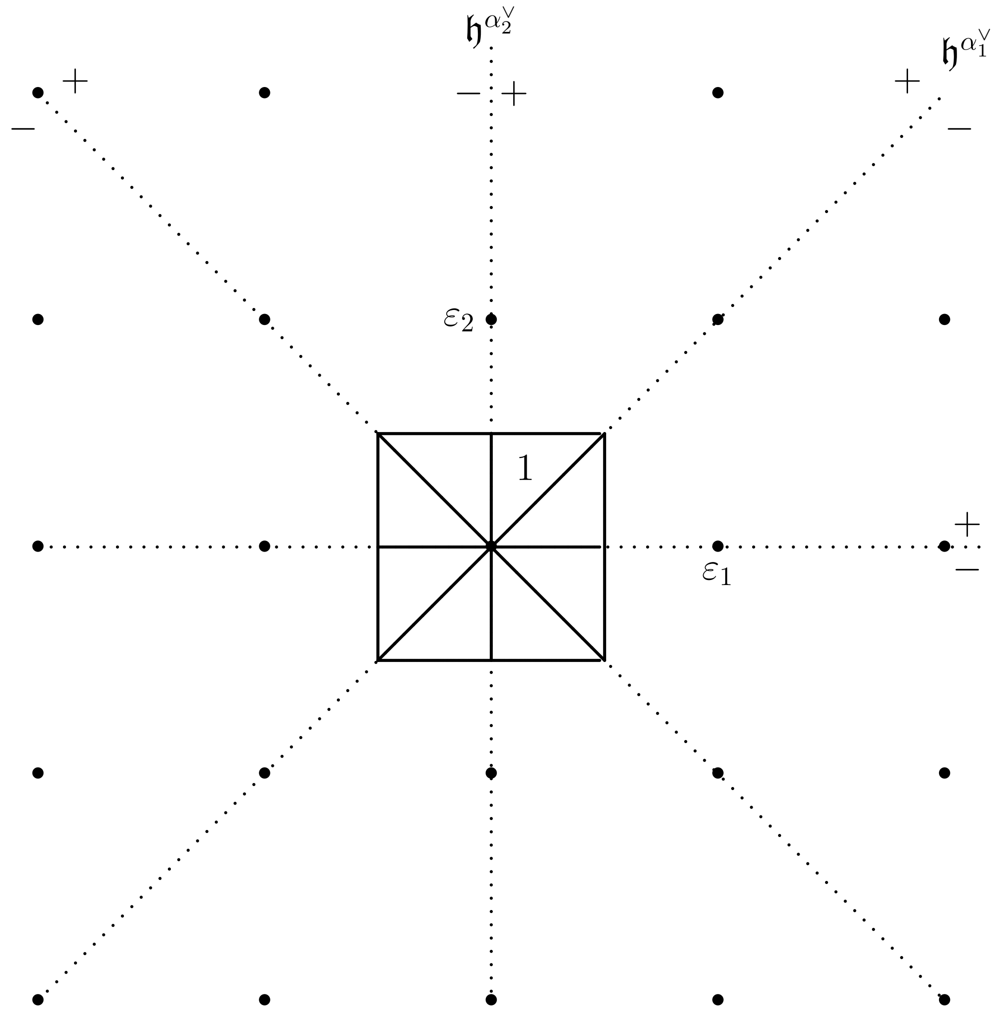

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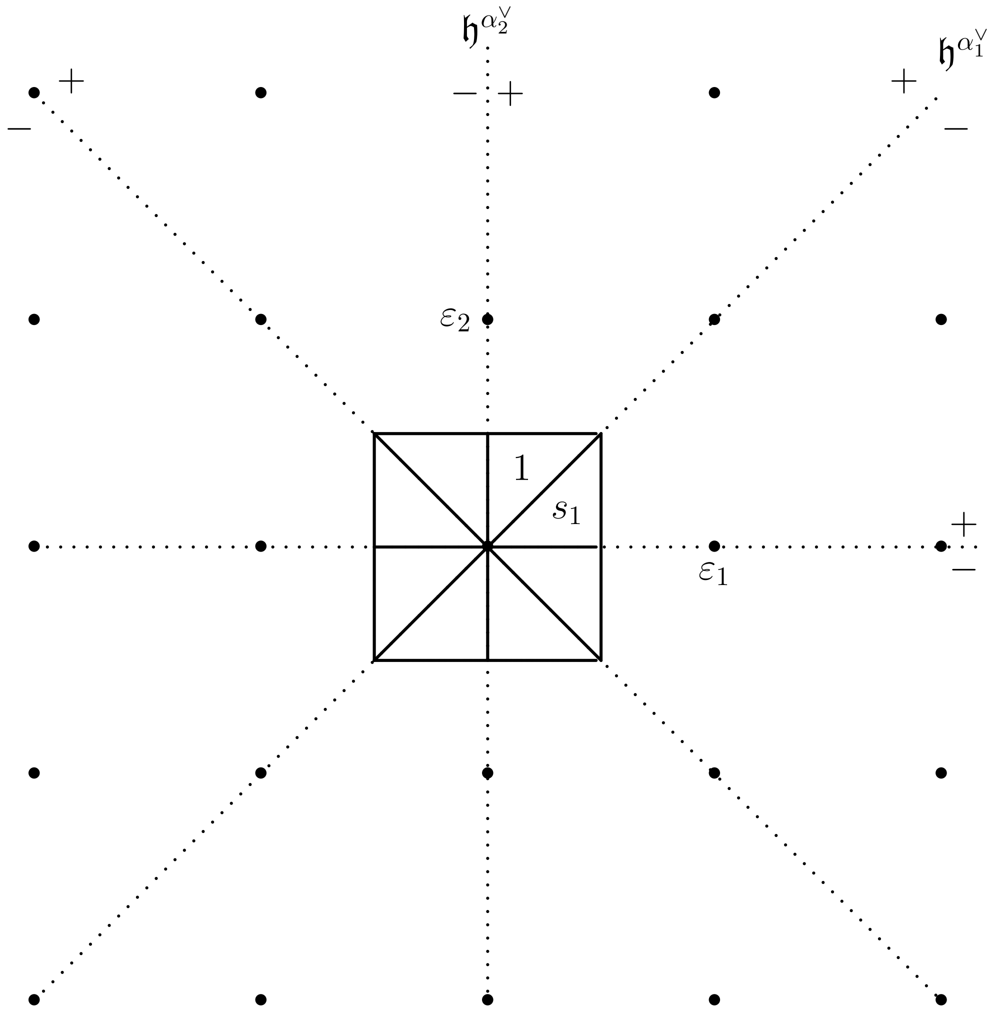

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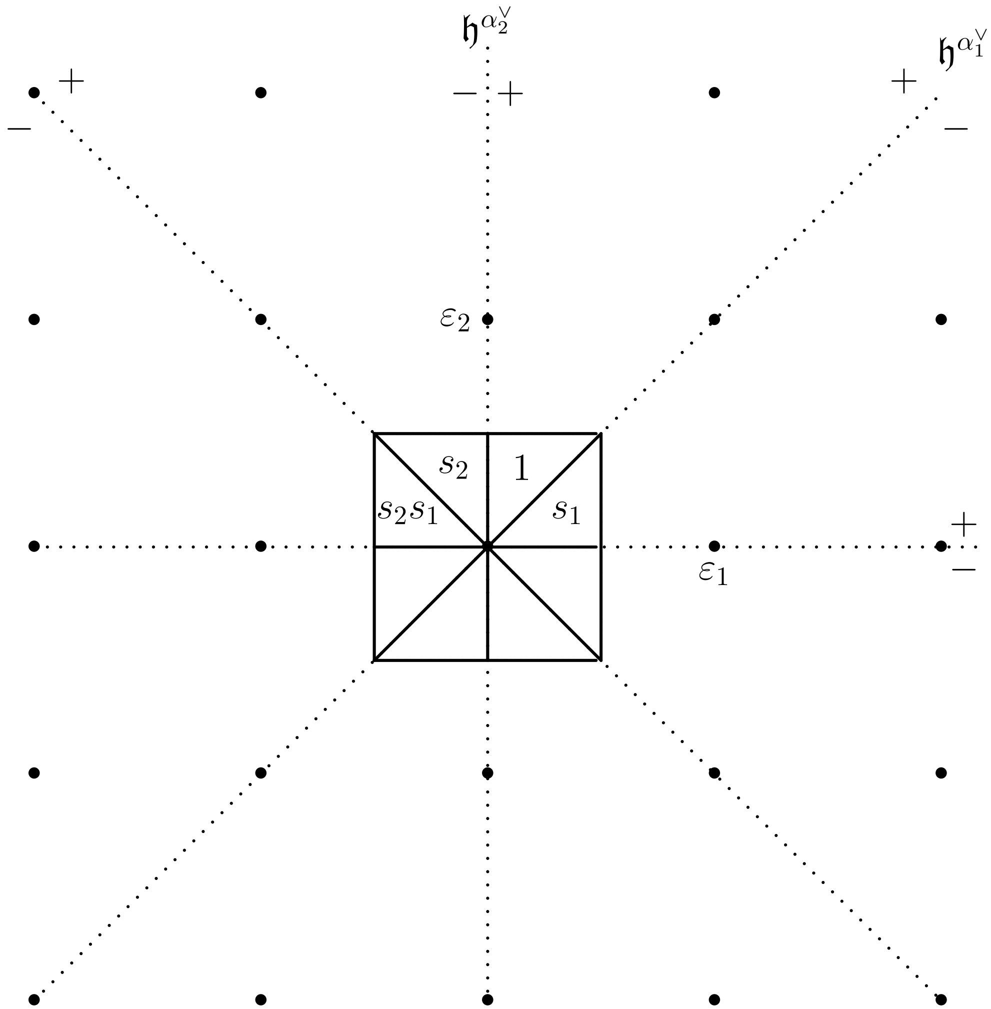

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s_2 is reflection in $\mathfrak{h}^{\alpha_2^\vee}$

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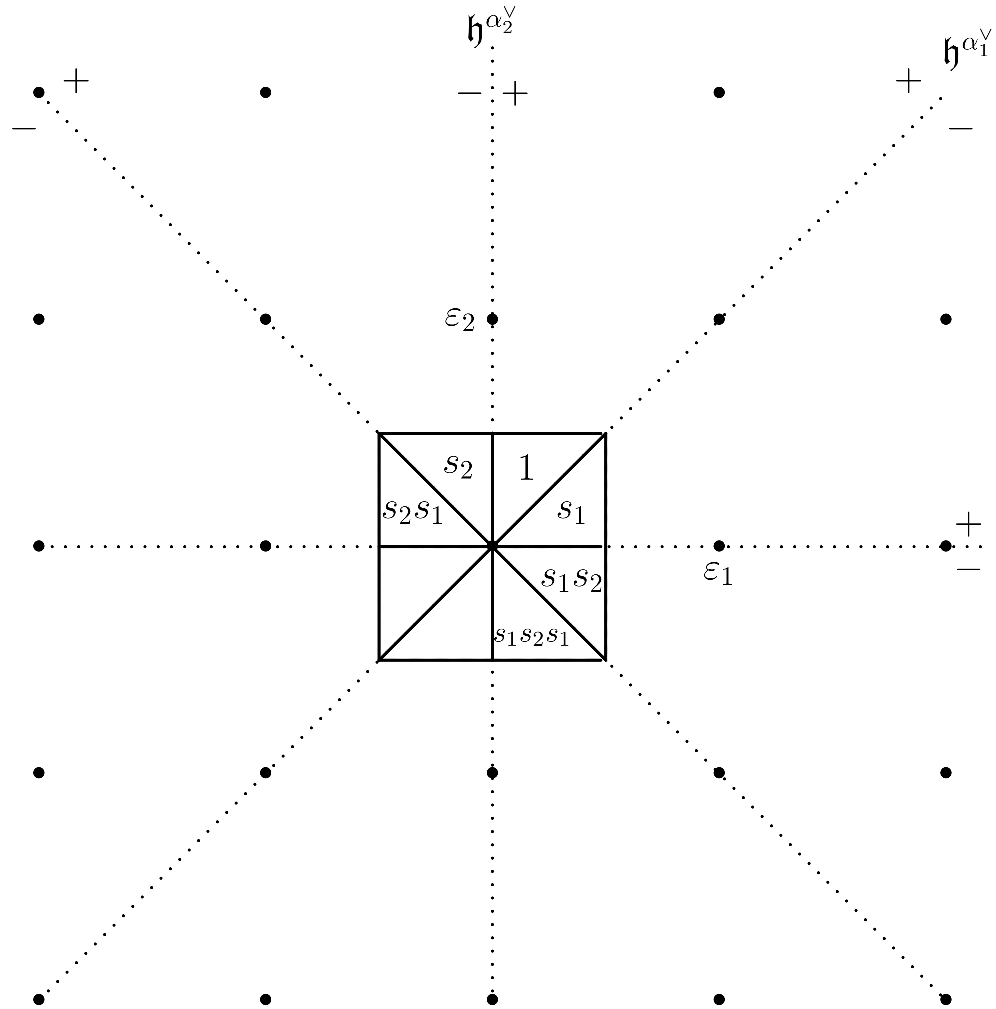

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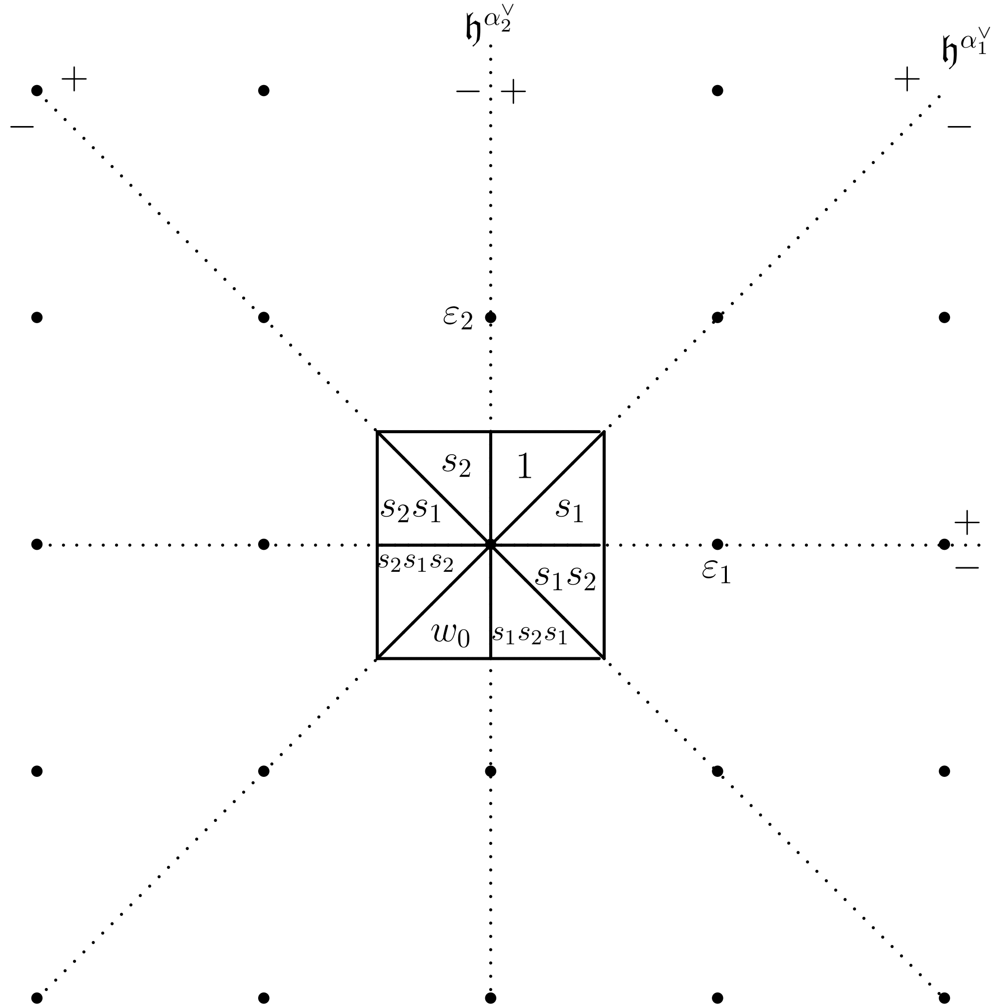
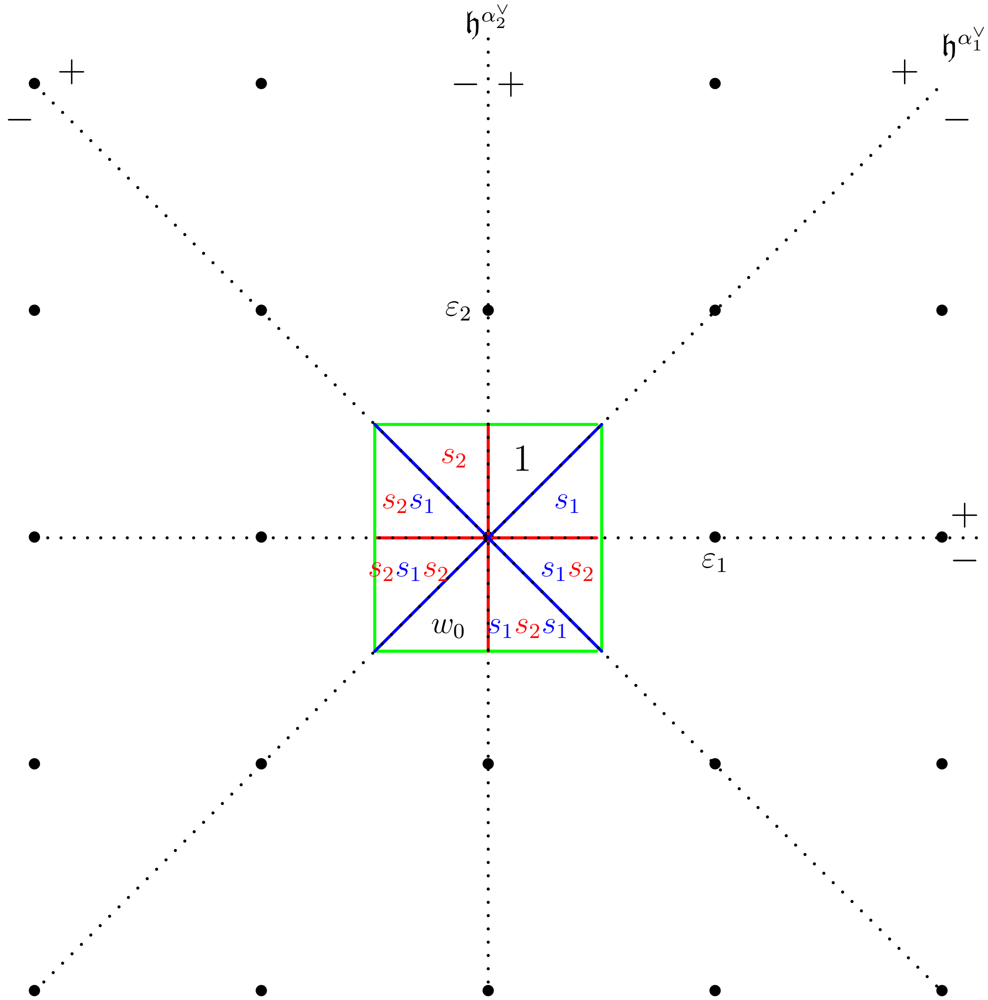


$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2\}$$

$$W_0 = \left\langle s_1, s_2 \;\middle|\; \begin{matrix} s_i^2 = 1, \\ s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \end{matrix} \right\rangle$$

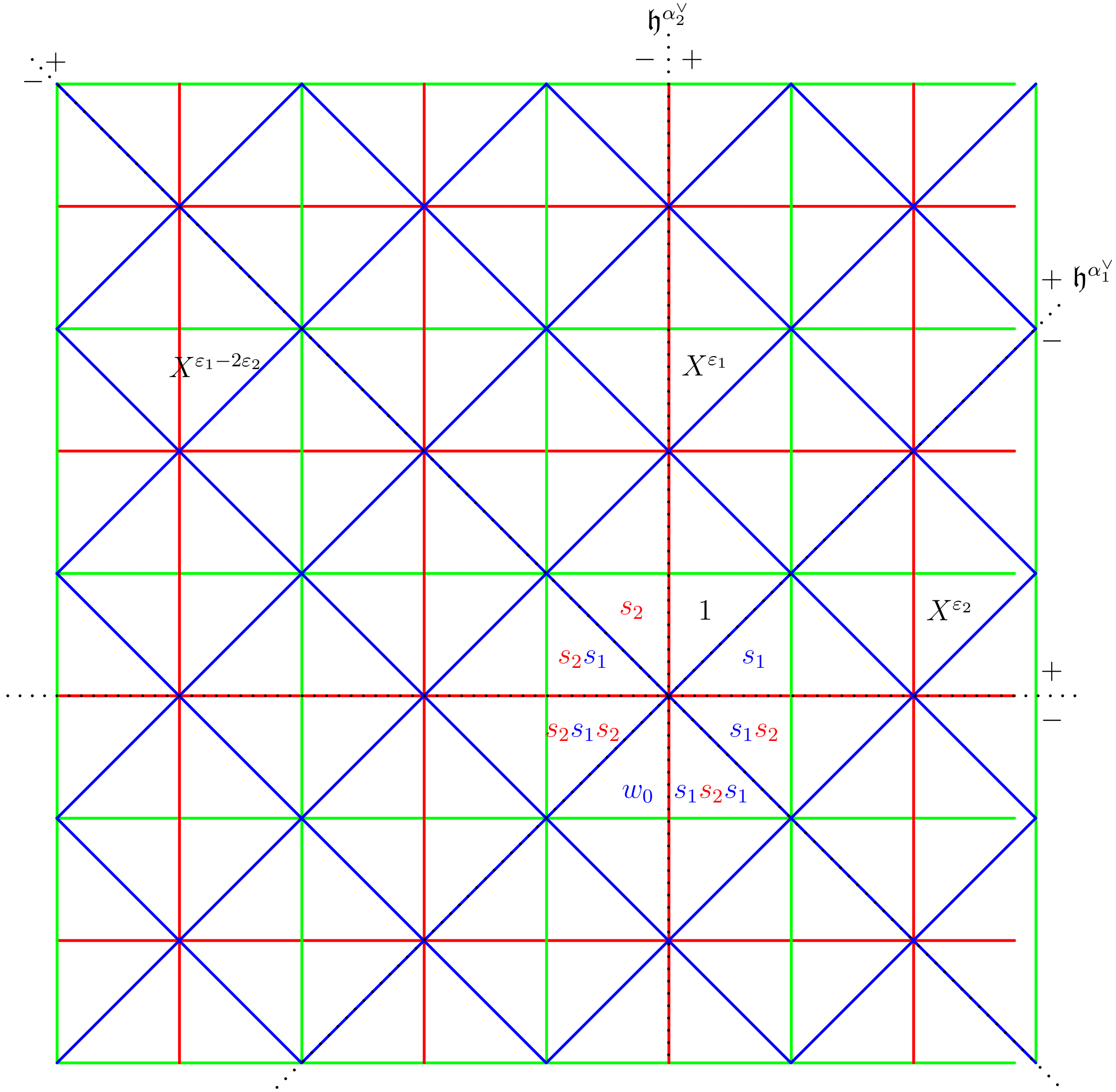
Compact Lie group $= (W_0, \mathfrak{h}^*_{\mathbb{Z}}) = W_0 \ltimes \mathfrak{h}^*_{\mathbb{Z}}$



$W = W_0 \ltimes \mathfrak{h}^*_{\mathbb{Z}}$

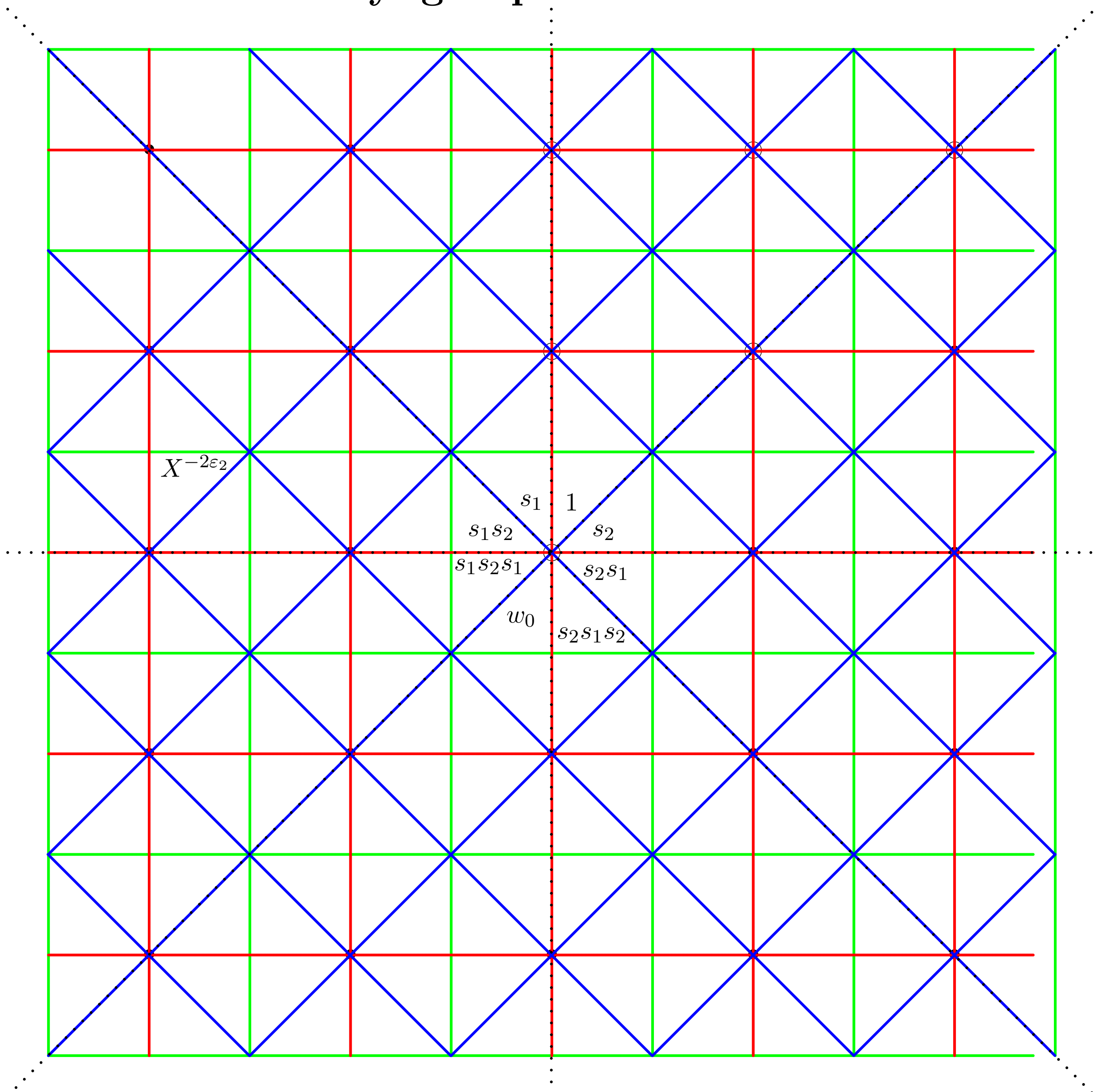
is the *affine Weyl group*

Affine Weyl group



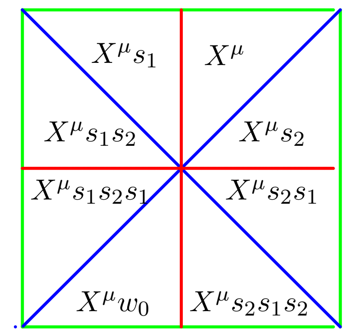
$$W = W_0 \ltimes \mathfrak{h}^*_{\mathbb{Z}} = \{X^\mu w\}$$

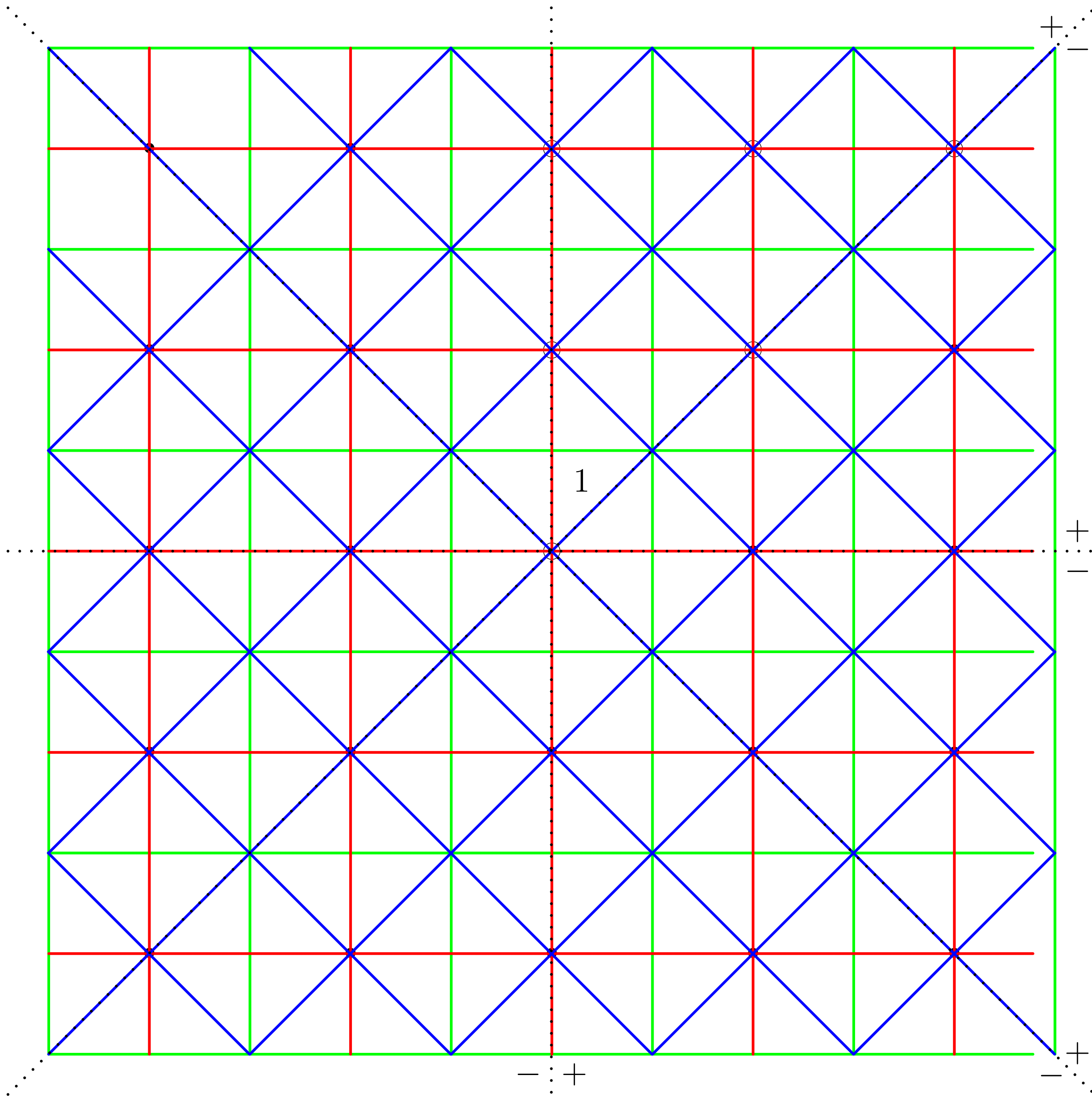
The affine Weyl group



$$W = W_0 \ltimes \mathfrak{h}^*_{\mathbb{Z}}$$

$$W = \{ X^\mu w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}},\ w \in W_0 \}$$





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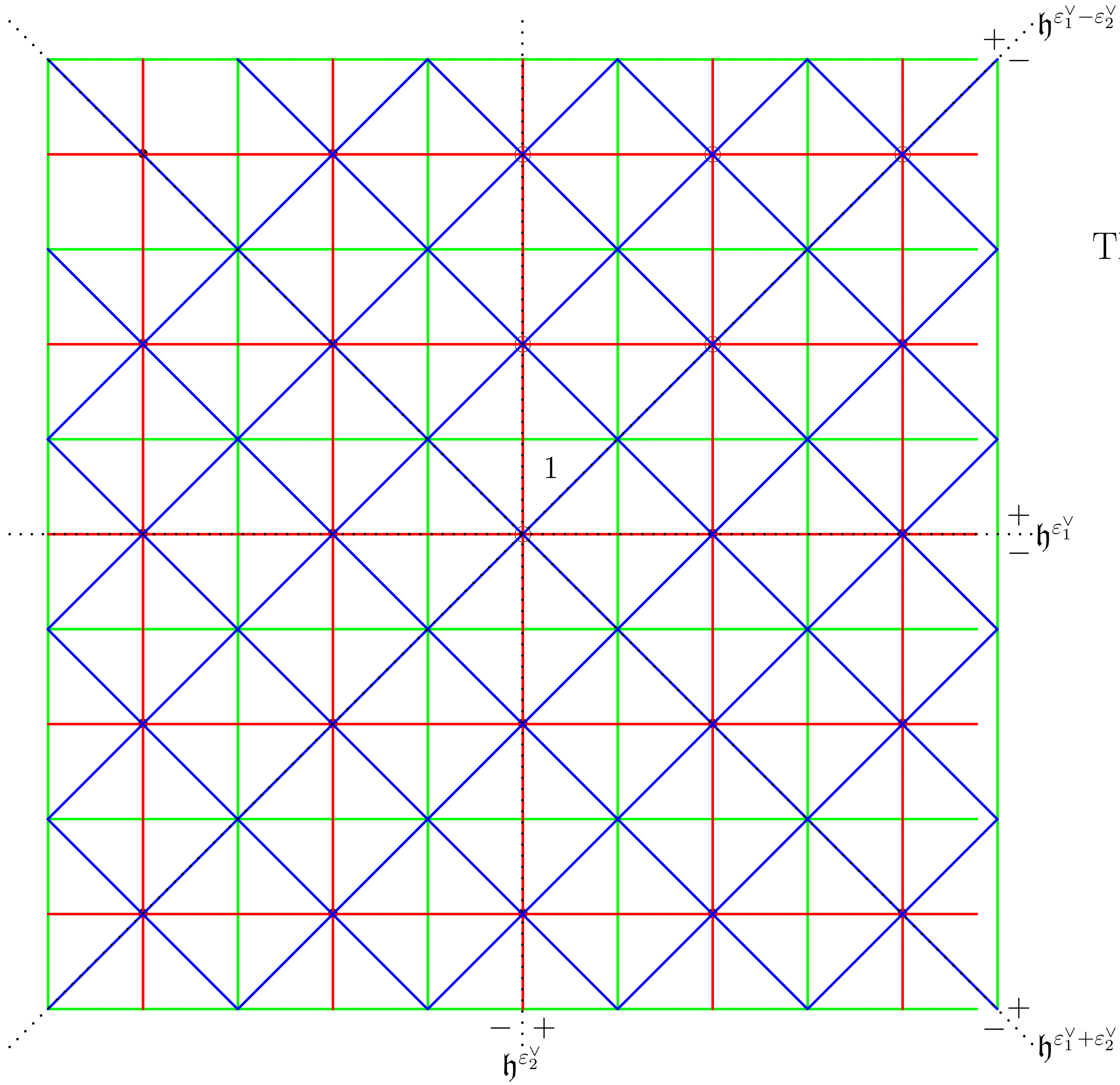

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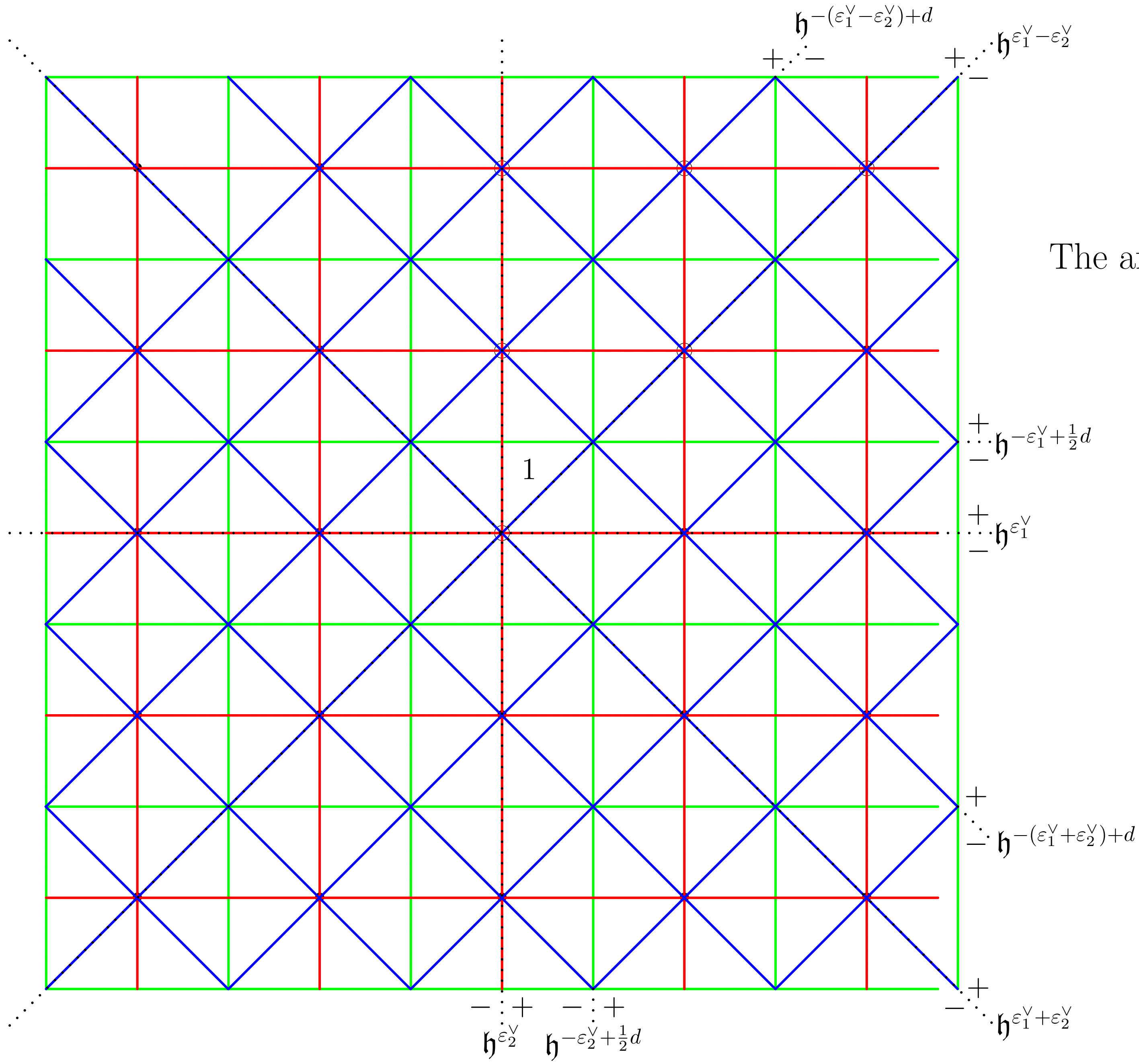

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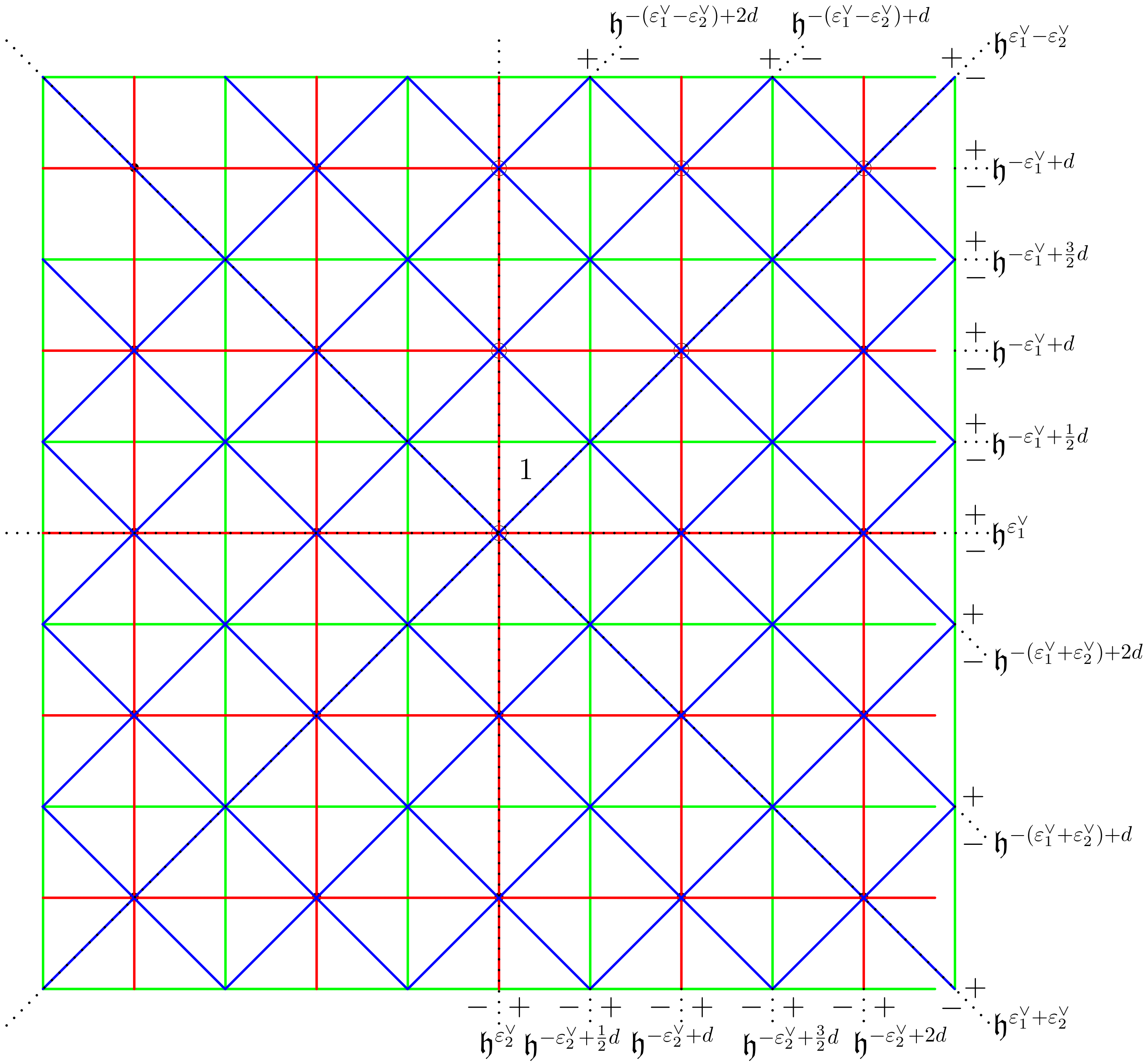

The affine Weyl

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Theorem (Ram-Yip)

Let $\lambda \in P^+$ (i.e. λ is a partition).

Let p_λ be a minimal length path to the λ-octagon.

The Macdonald polynomial P_λ *is given by*

$$P_\lambda = \sum_{w \in W_0} \sum_{\substack{\text{foldings } p \\ \text{of } wp_\lambda}} t_{i_1}^{\frac{1}{2}} \cdots t_{i_\ell}^{\frac{1}{2}} \left(\prod_{k \in F^+(p)} f_k^+ \right) \left(\prod_{k \in F^-(p)} f_k^- \right) X^{\mathrm{wt}(p)} \, t_{j_1}^{\frac{1}{2}} \cdots t_{j_r}^{\frac{1}{2}}$$

Parsing the formula

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$t_0^{\frac{1}{2}}, t_1^{\frac{1}{2}}, t_2^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_2^{\frac{1}{2}}, q^{\frac{1}{2}},$ are variables (elements of my base ring)

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$$X^\mu = X^{\mu_1\varepsilon_1+\cdots+\mu_n\varepsilon} = (X^{\varepsilon_1})^{\mu_1}\cdots(X^{\varepsilon_n})^{\mu_n} = x_1^{\mu_1}\cdots x_n^{\mu_n}$$

for $\mu = \mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n \in \mathfrak{h}_{\mathbb{Z}}^*$,

$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\varepsilon_1,\ldots,\varepsilon_n\}$

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Let $\lambda \in P^+$ (i.e. λ is a partition).

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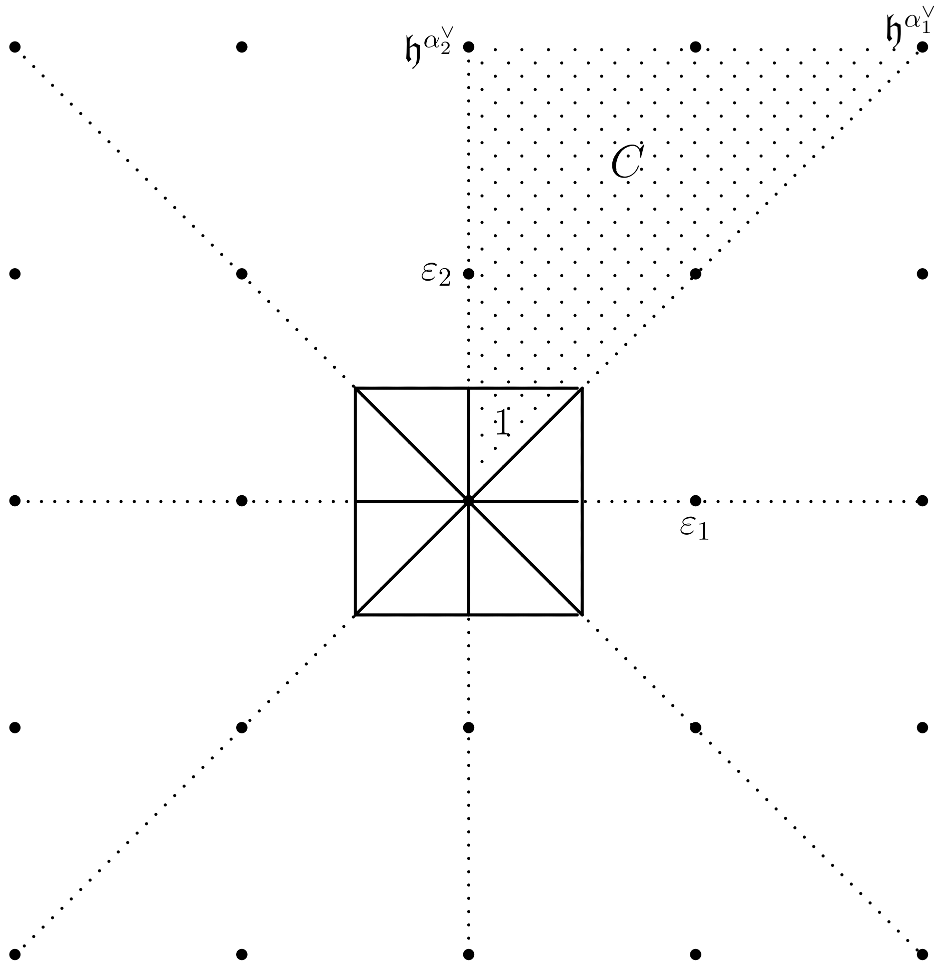
Partitions λ are elements of P^+



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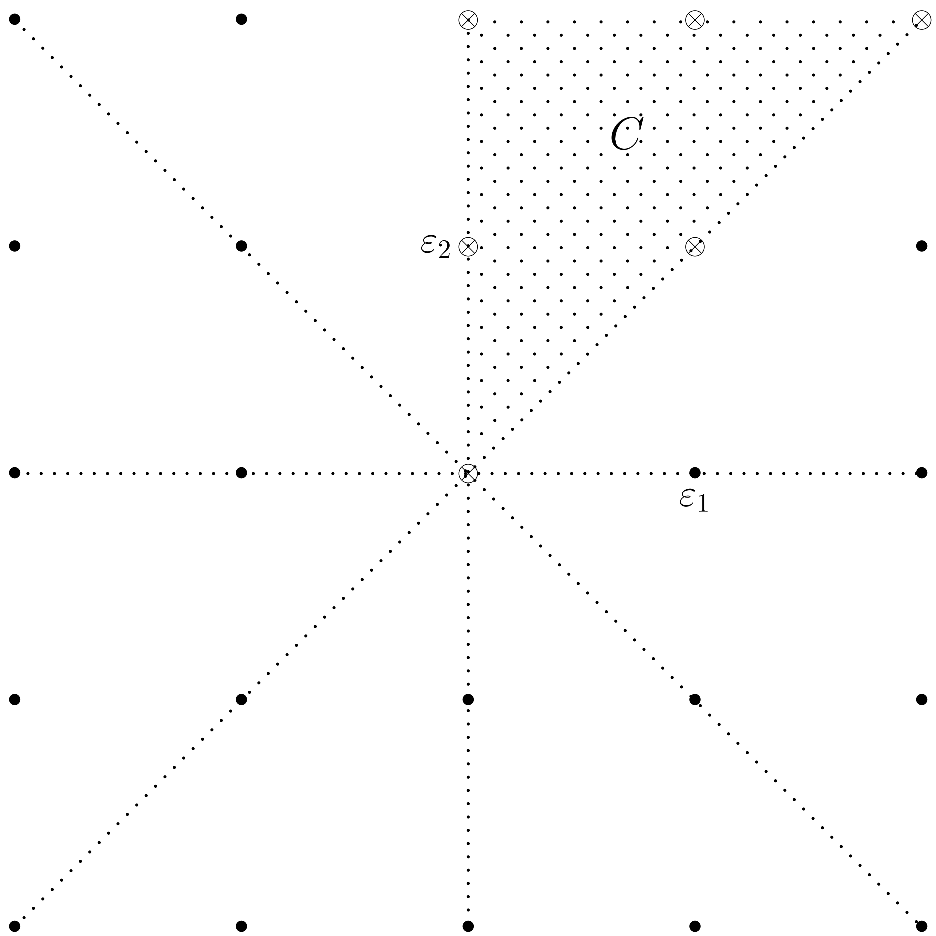

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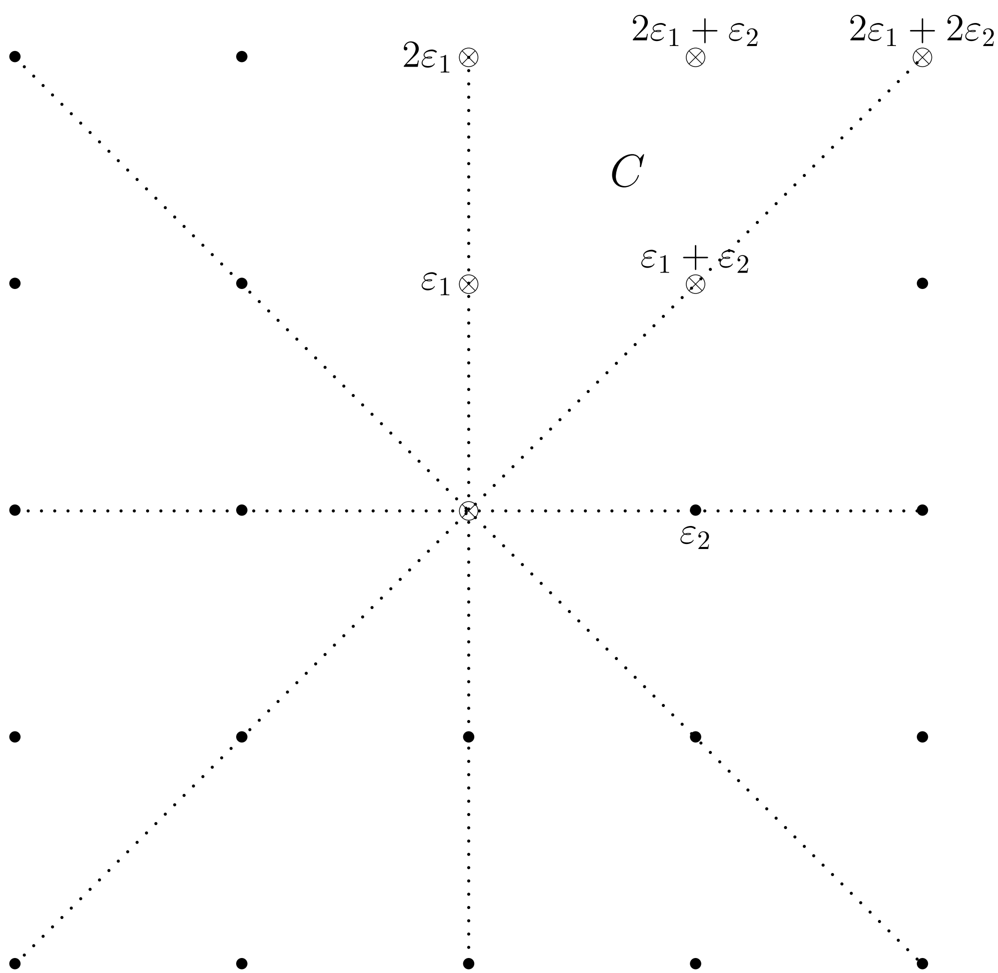

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$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

$n = 2$ for this picture

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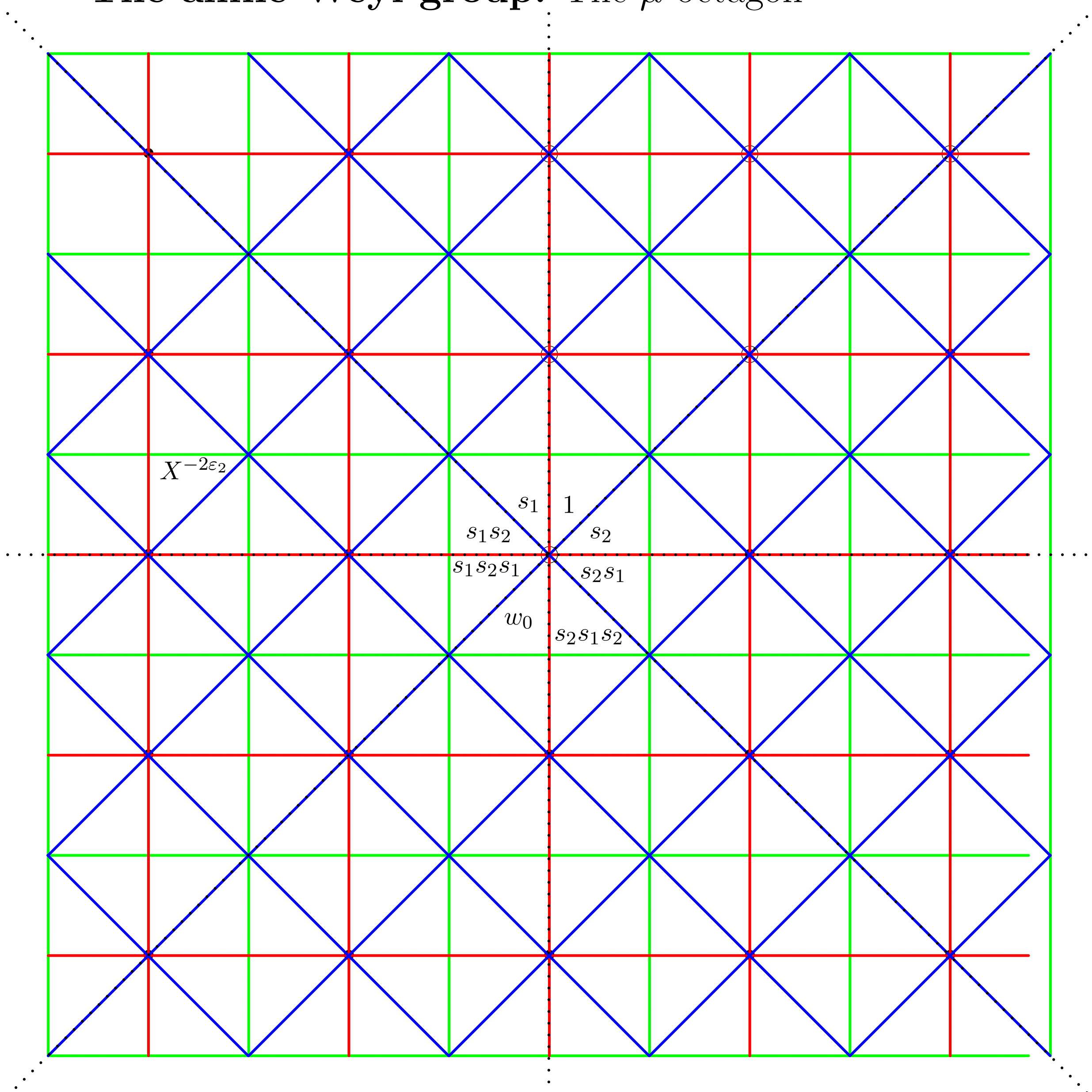
The affine Weyl group: The μ-octagon



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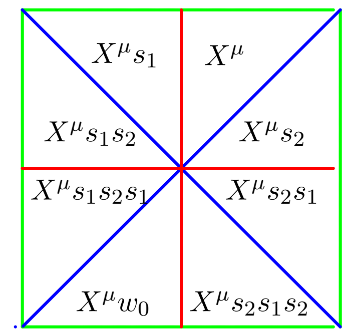

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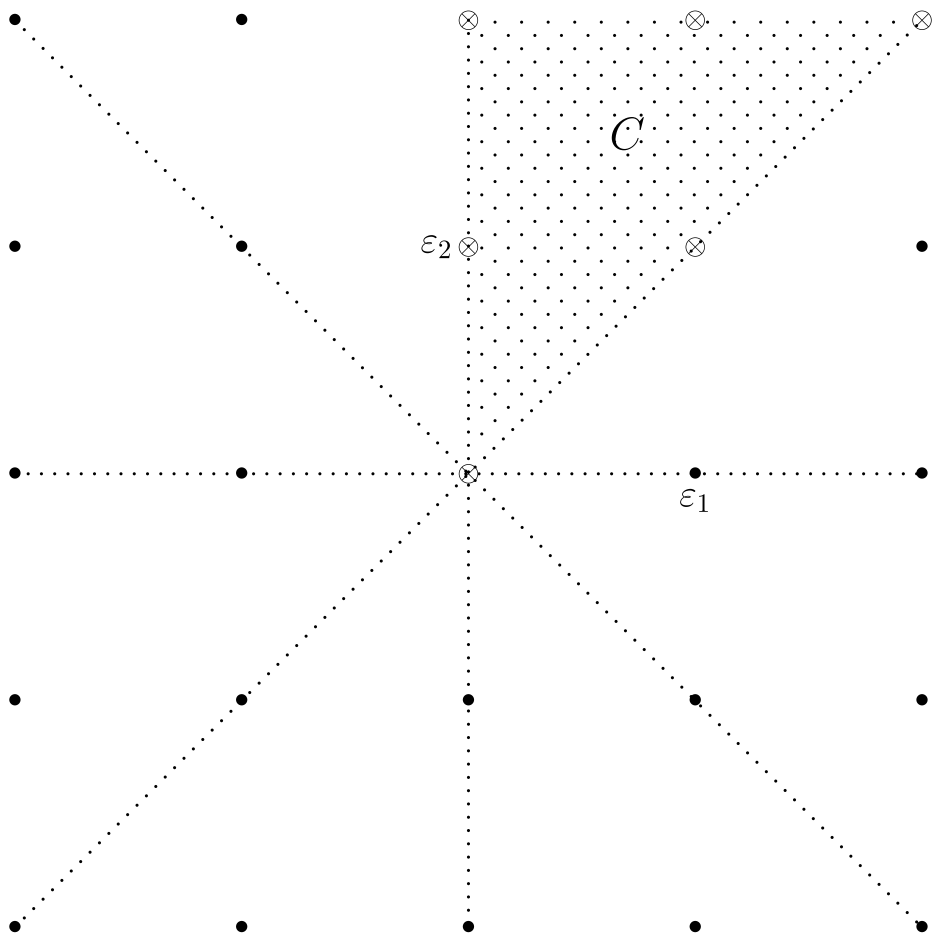

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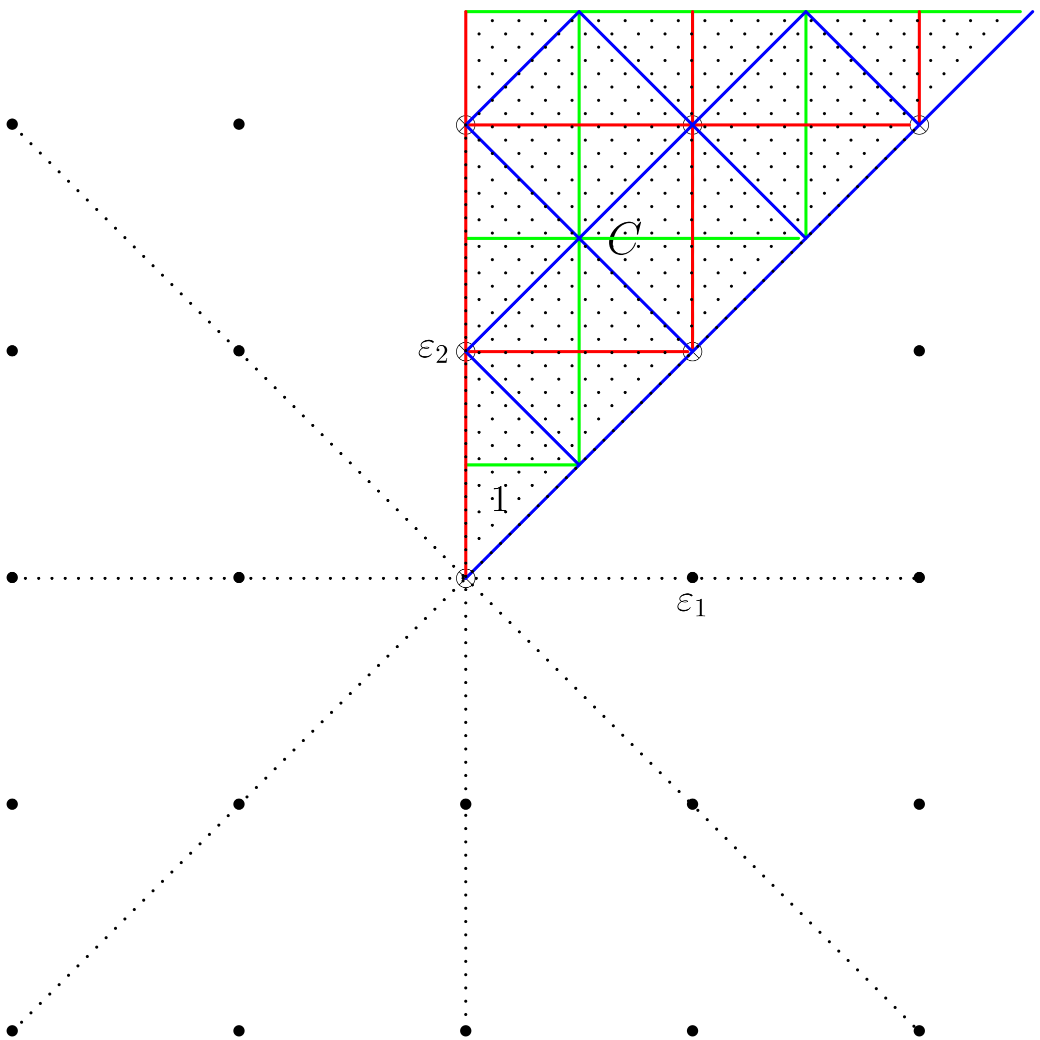

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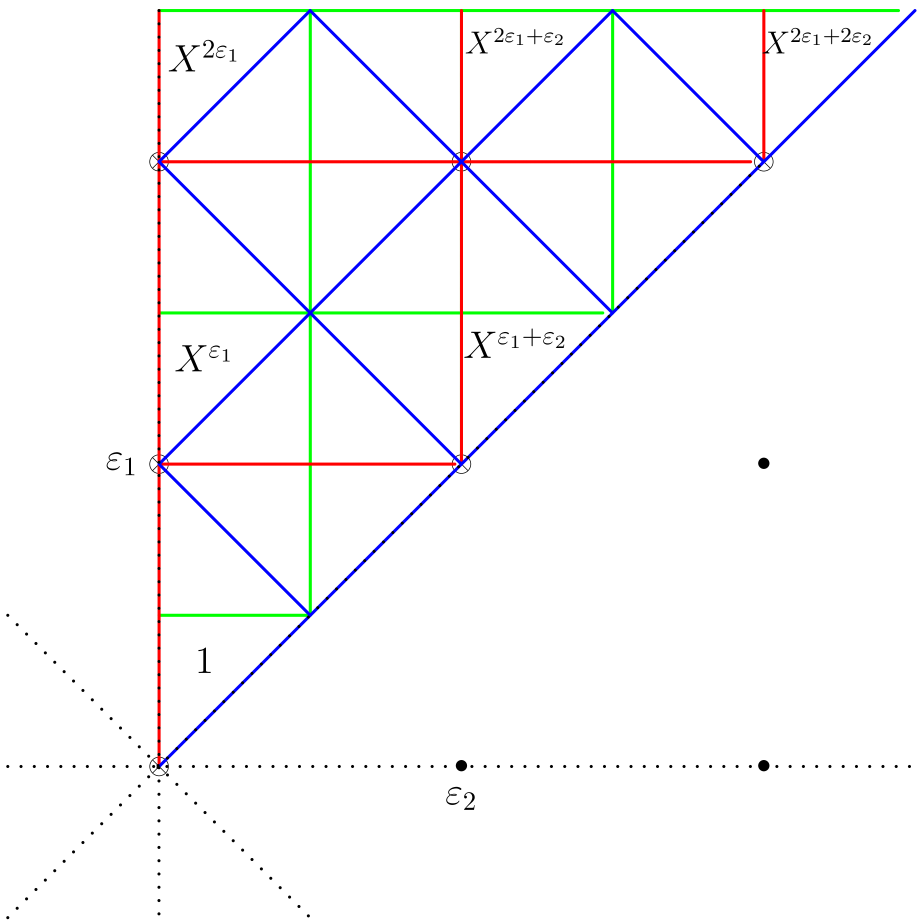

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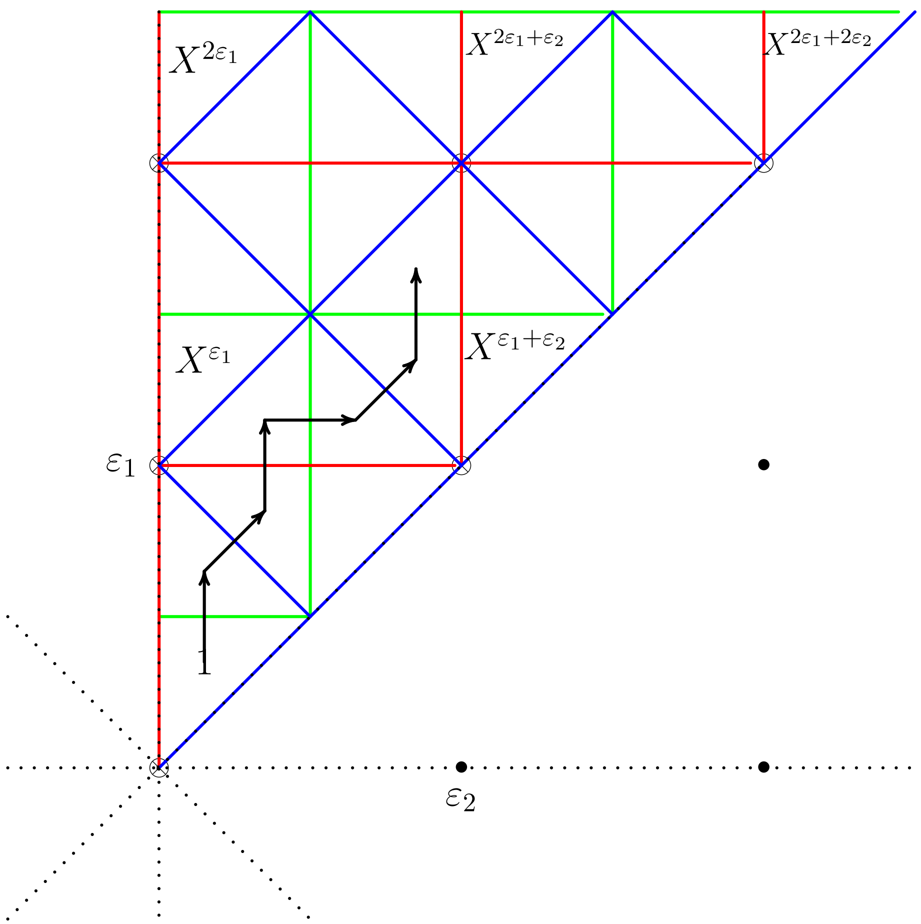
The path p_λ for $\lambda = 2\varepsilon_1 + \varepsilon_2$



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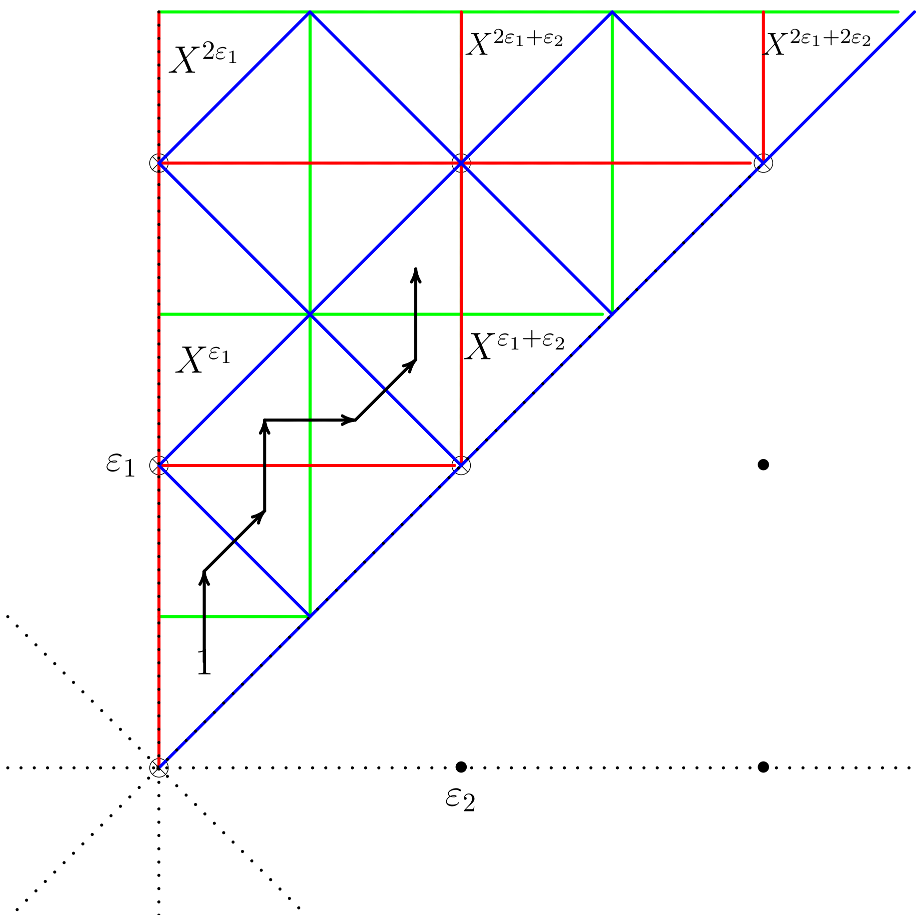

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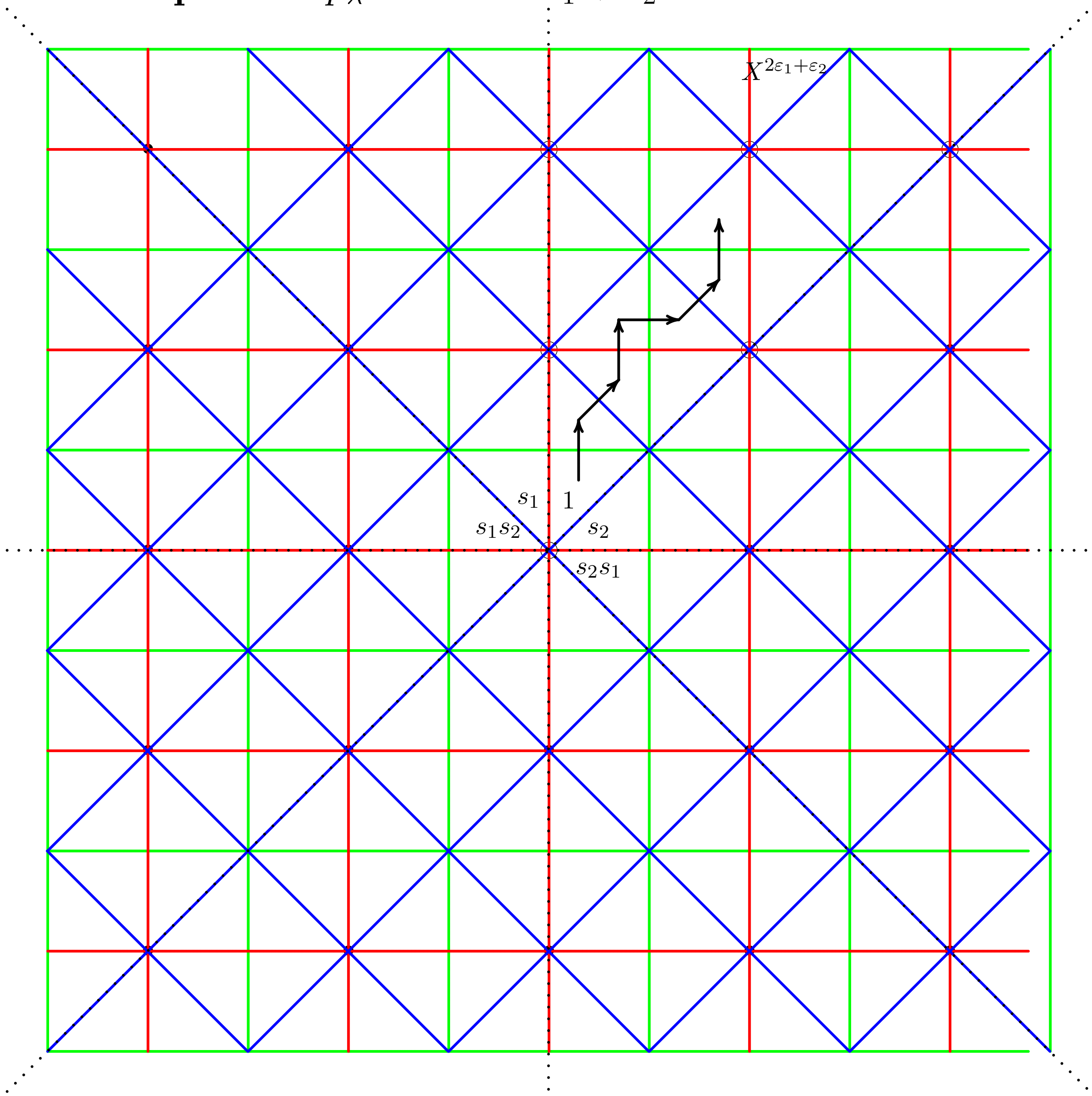
The paths wp_λ for $\lambda = 2\varepsilon_1 + \varepsilon_2$



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The path p_λ

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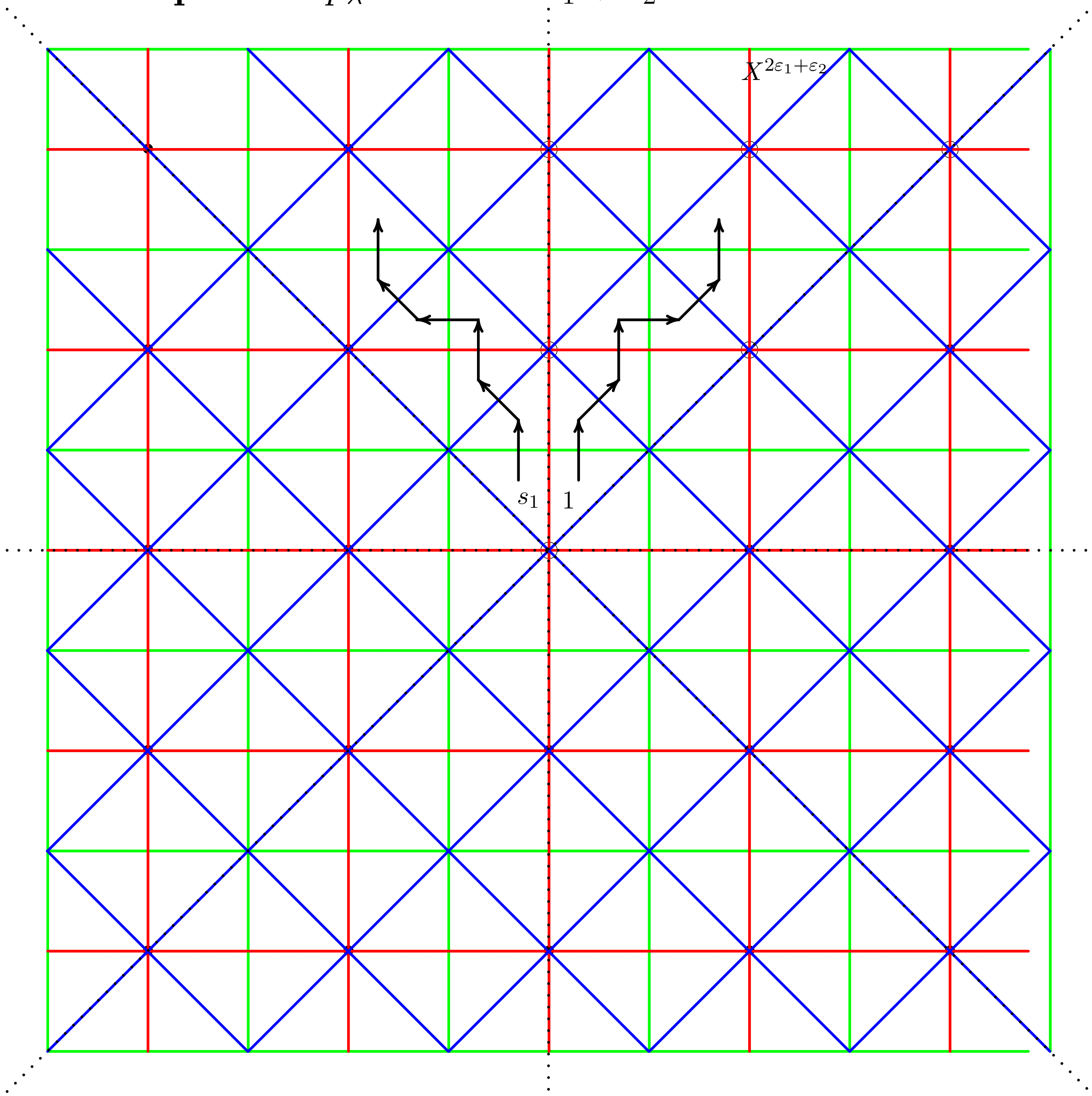

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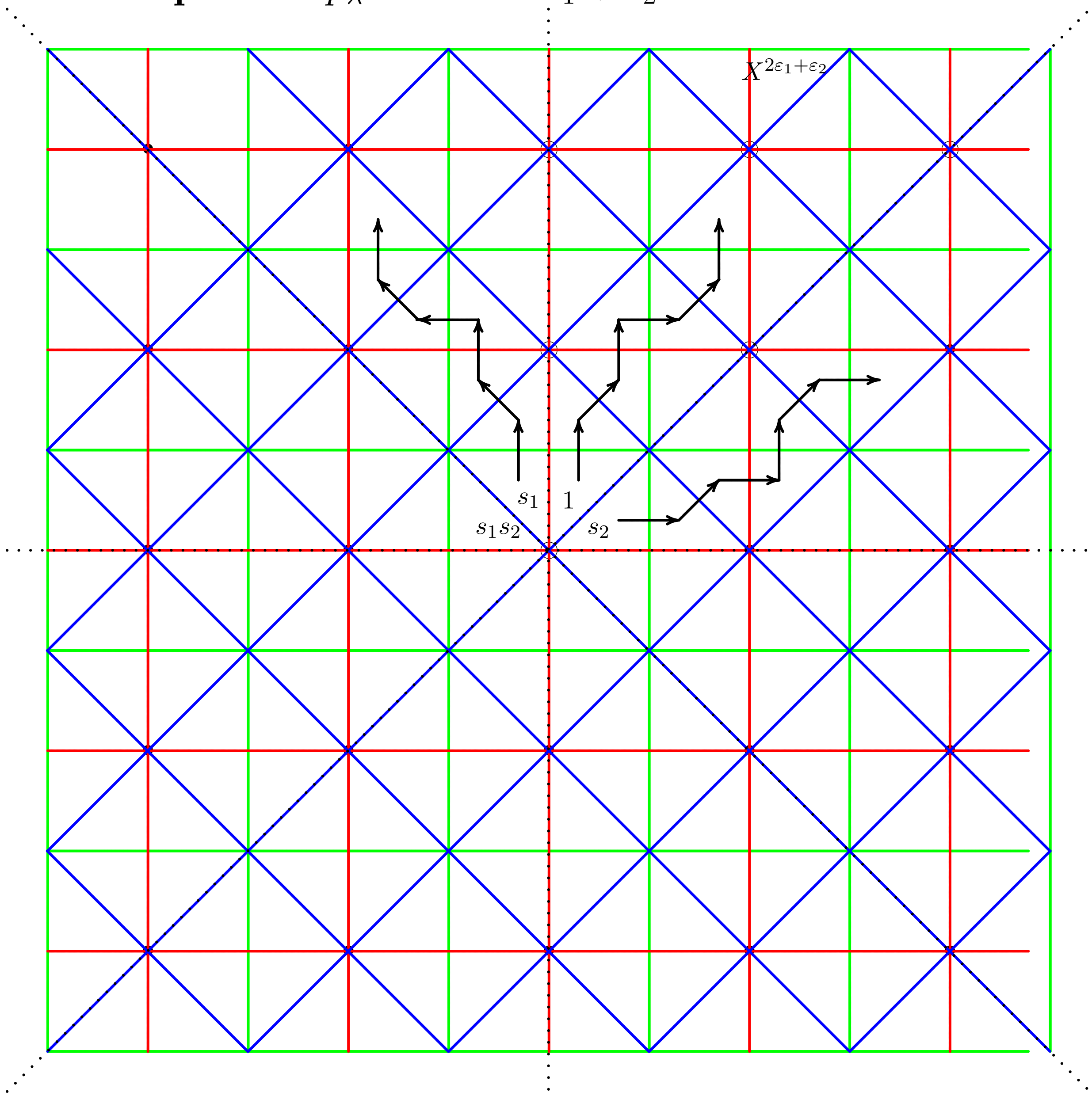
p_λ and $s_1 p_\lambda$

The paths wp_λ for $\lambda = 2\varepsilon_1 + \varepsilon_2$



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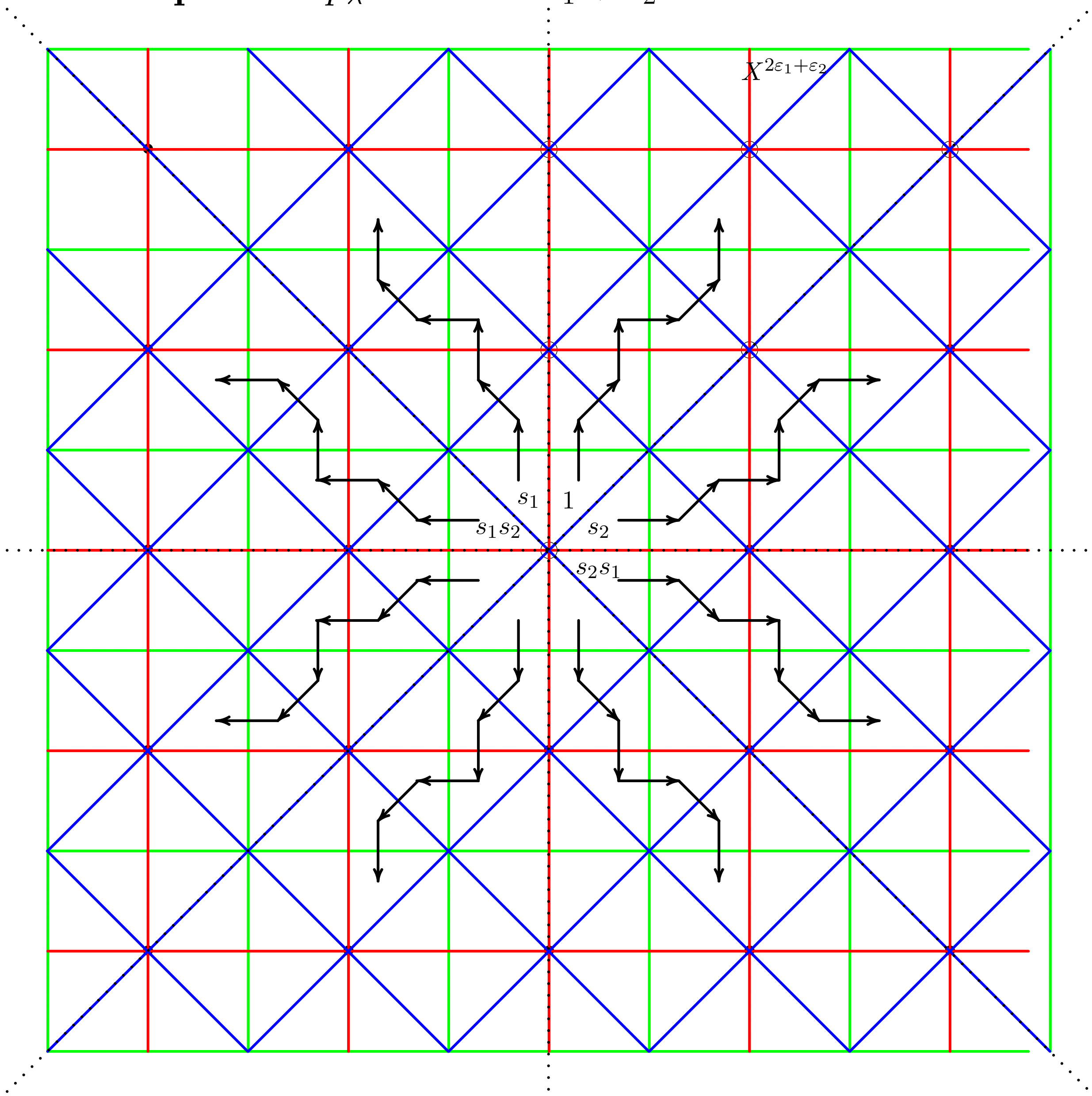
p_λ, $s_1 p_\lambda$, and $s_2 p_\lambda$

The paths wp_λ for $\lambda = 2\varepsilon_1 + \varepsilon_2$



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wp_λ, for all $w \in W_0$

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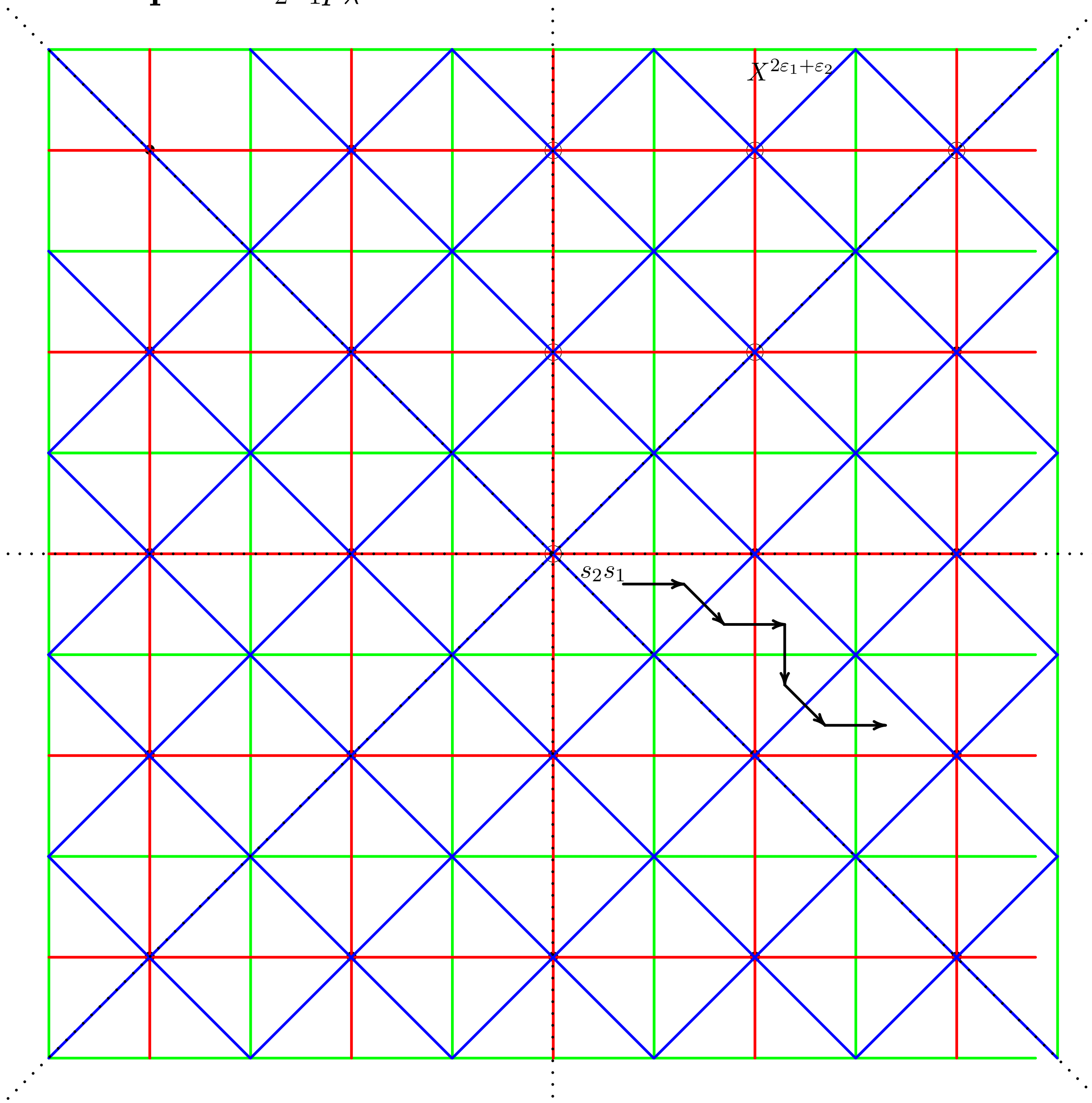
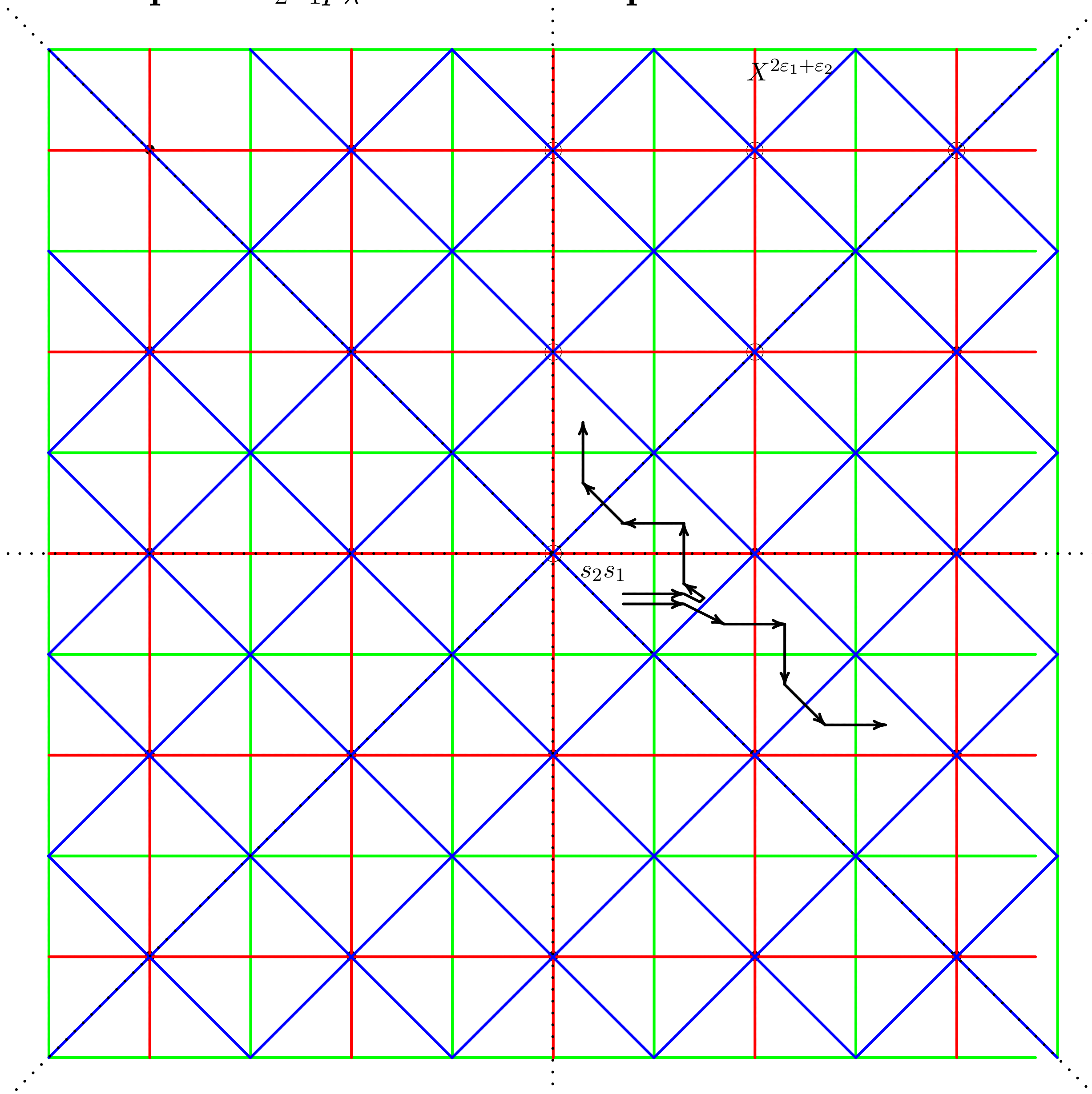
The path $s_2 s_1 p_\lambda$



foldings p of wp_λ

Here $wp_\lambda = s_2 s_1 \, p_\lambda$

The path $s_2 s_1 p_\lambda$ folded at step 2



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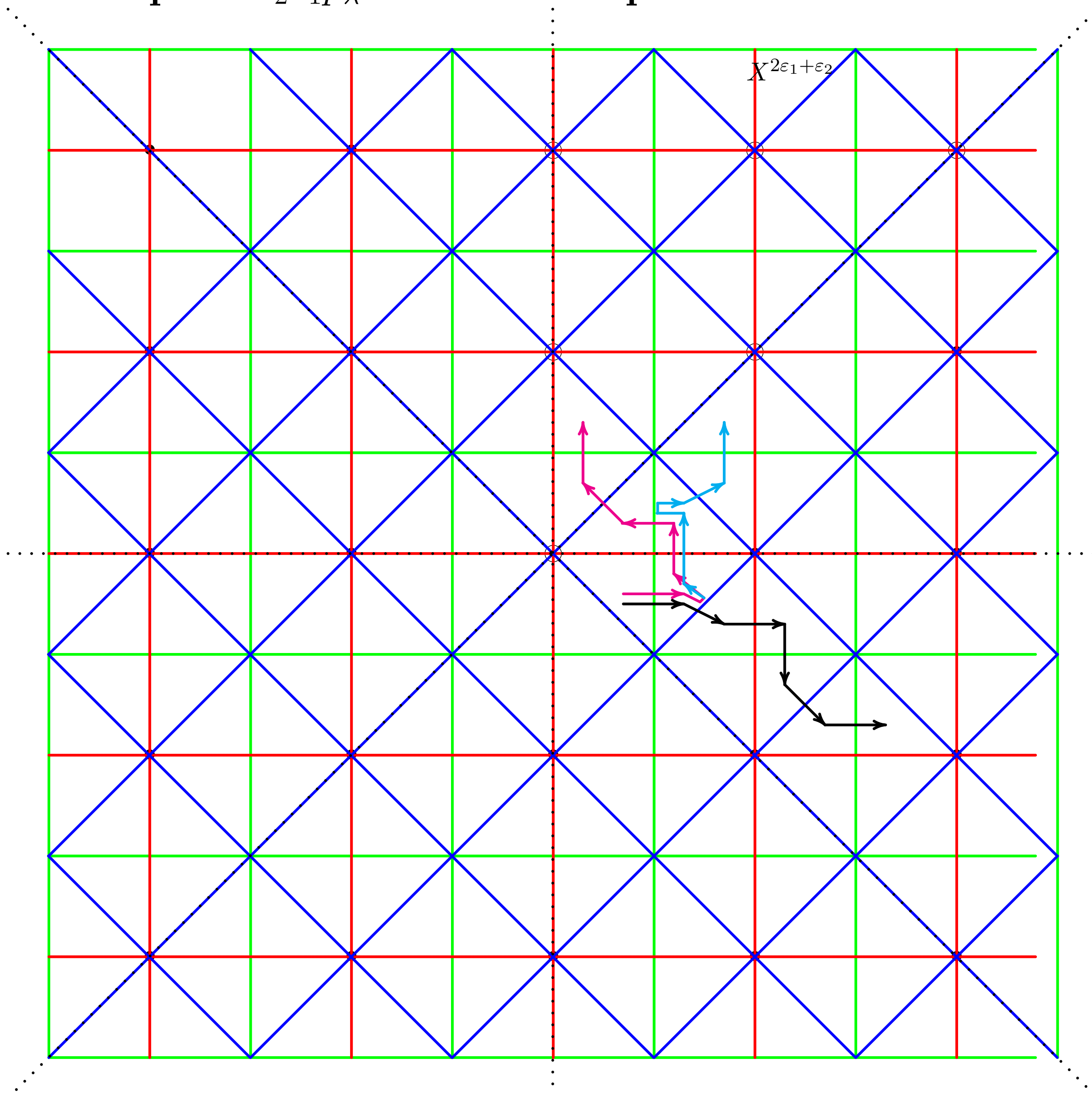
The path $s_2 s_1 p_\lambda$ folded at steps 2 and 5



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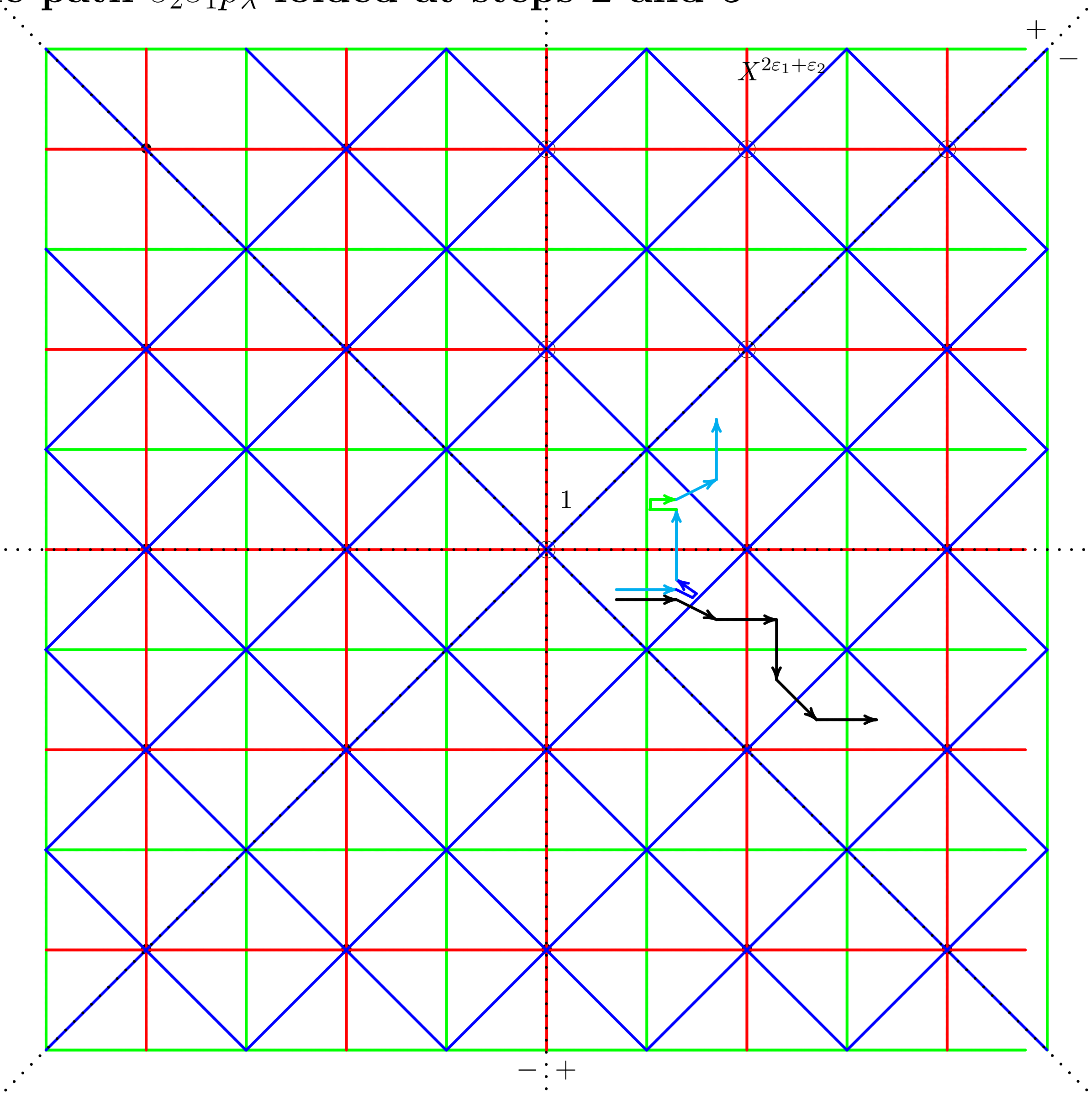

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$$P_\lambda = \sum_{w \in W_0} \sum_{\substack{\text{foldings } p \\ \text{of } wp_\lambda}} t_{i_1}^{\frac{1}{2}} \cdots t_{i_\ell}^{\frac{1}{2}} \left(\prod_{k \in F^+(p)} f_k^+ \right) \left(\prod_{k \in F^-(p)} f_k^- \right) X^{\mathrm{wt}(p)} \, t_{j_1}^{\frac{1}{2}} \cdots t_{j_r}^{\frac{1}{2}}$$

$F^+(p) = \{k \mid \text{the } k\text{th step of } p \text{ is a positive fold}\}$

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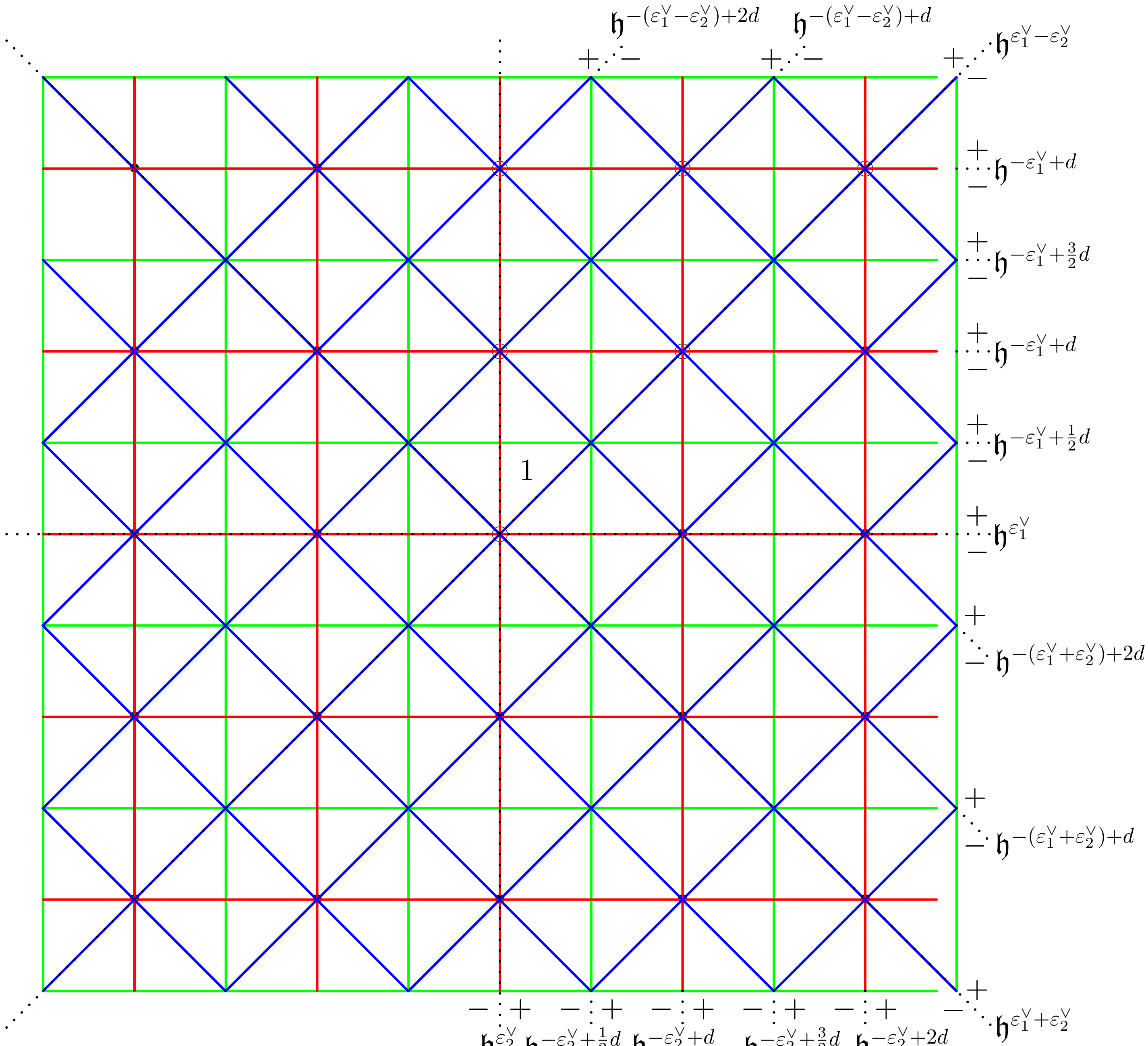
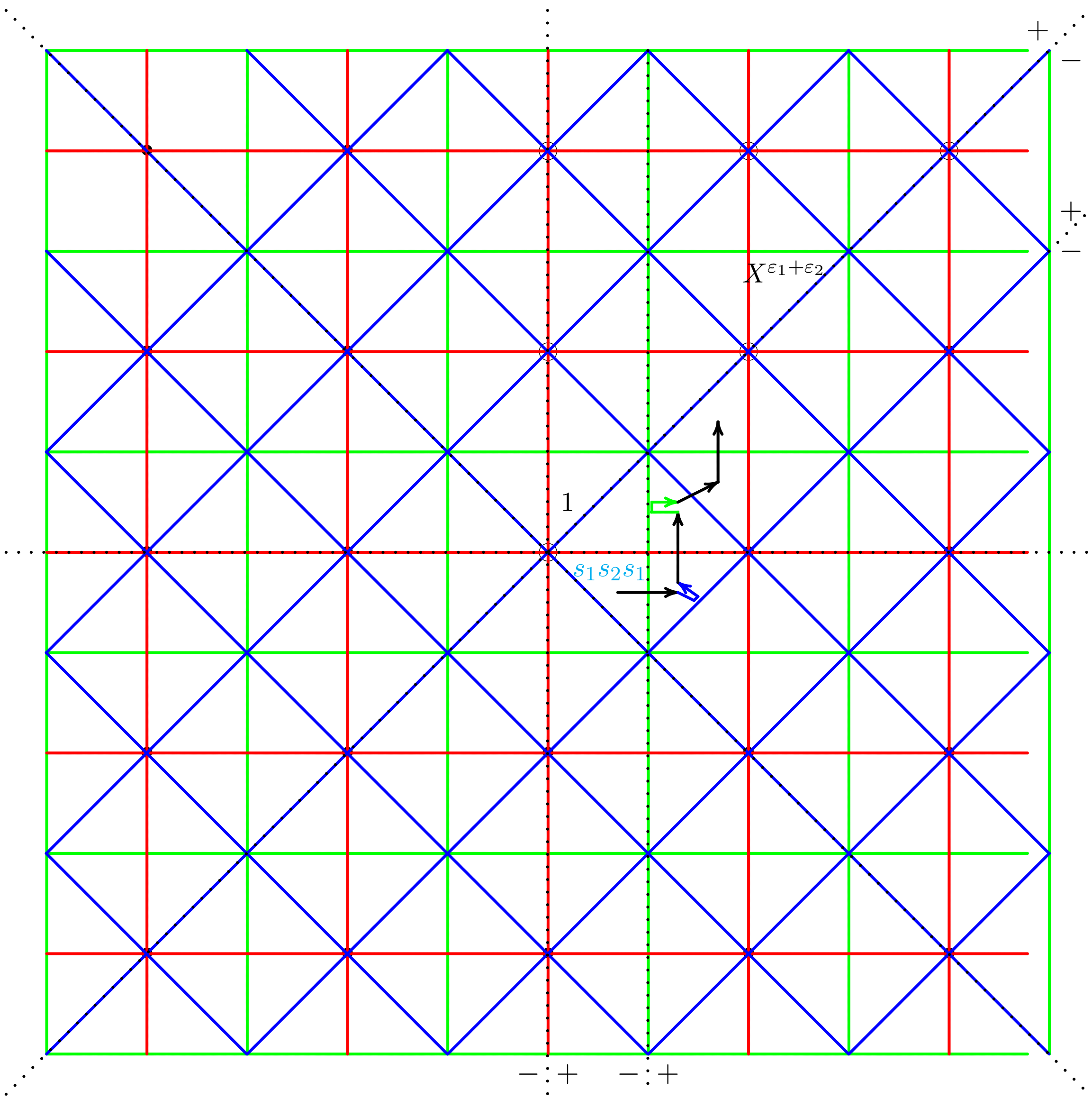


The affine Weyl group



$$F^+(p) = \{2, 5\}$$

$$F^-(p) = \emptyset$$

$$F^+(p) = \left\{ k \;\middle|\; \begin{array}{c}\text{the } k\text{th step of } p \\ \text{is a positive fold}\end{array} \right\}$$

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Parsing the formula

Let $\lambda \in P^+$ (i.e. λ is a partition).

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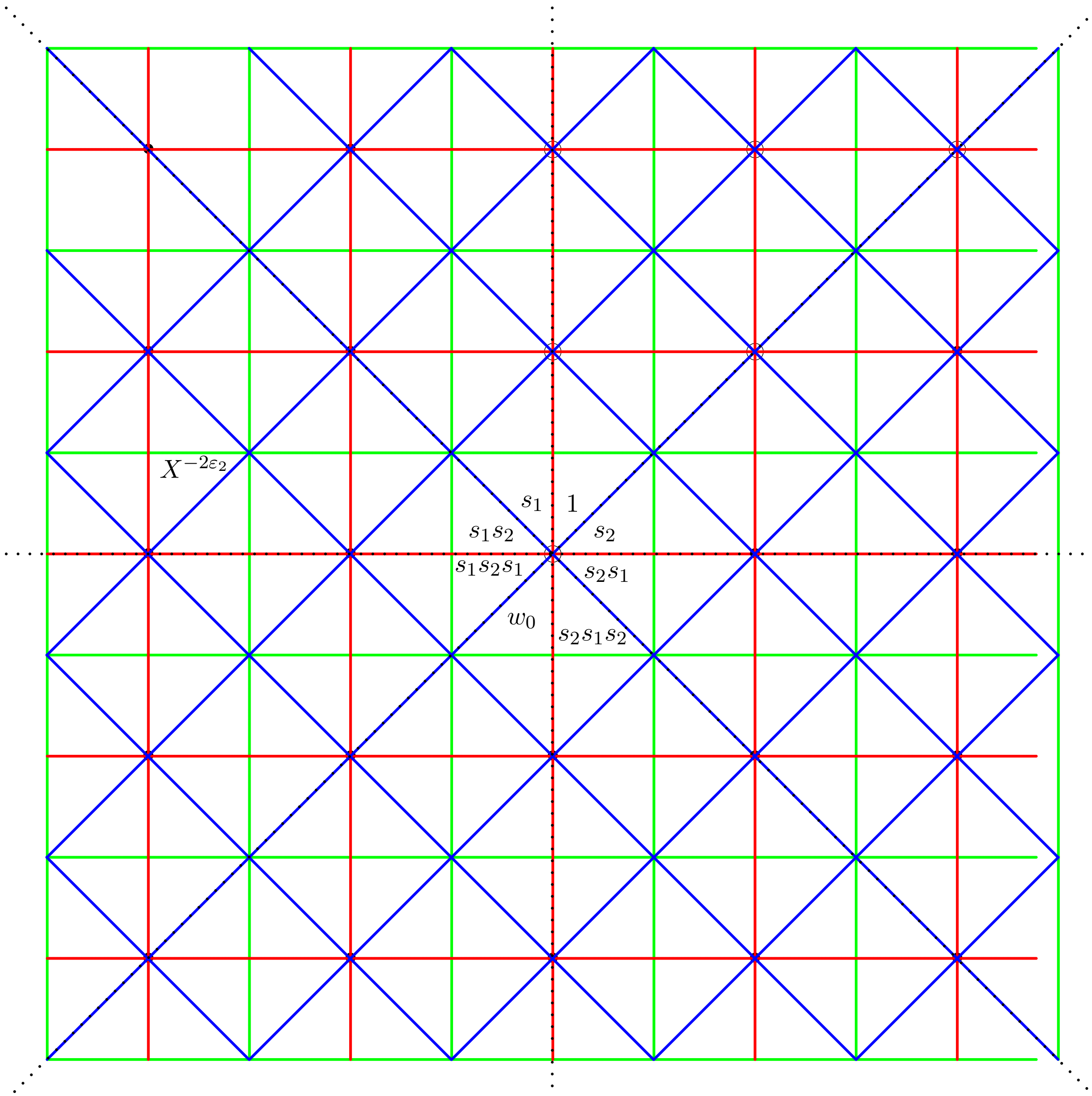
p begins at $s_{i_1} \cdots s_{i_\ell}$

p ends at $X^{\mathrm{wt}(p)} s_{j_1} \cdots s_{j_r}$



p begins at $s_{i_1} \dots s_{i_\ell}$

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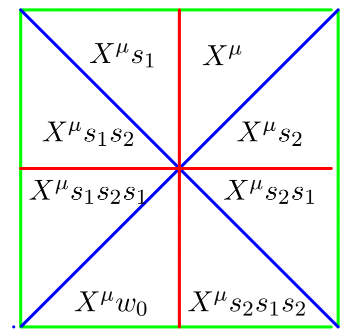

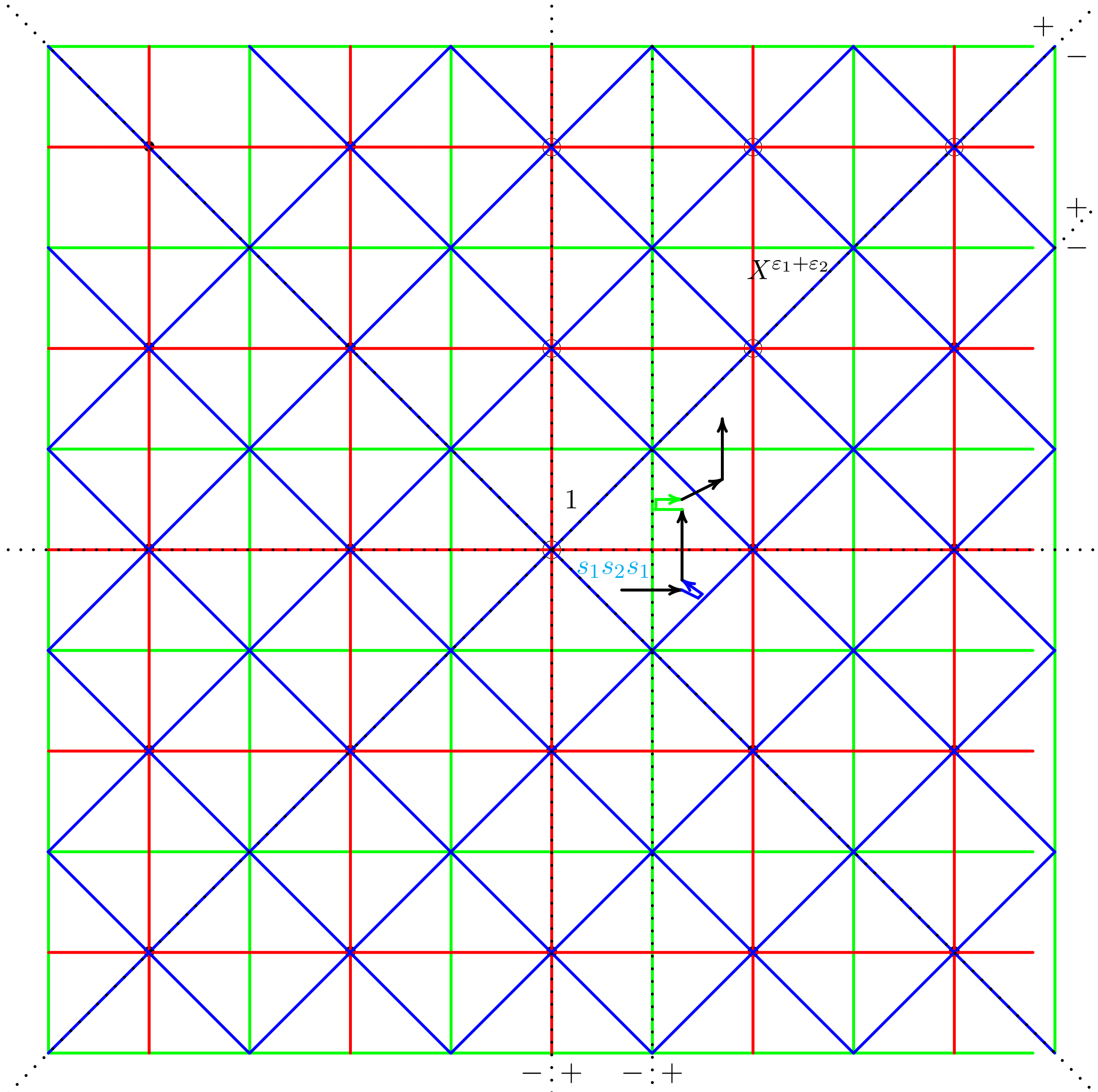

p begins at $s_{i_1}\dots s_{i_\ell}$

and ends at $X^{\mathrm{wt}(p)}s_{j_1}\cdots s_{j_r}$

$$t_{i_1}^{\frac{1}{2}}\cdots t_{i_\ell}^{\frac{1}{2}}\quad X^{\mathrm{wt}(p)}\, t_{j_1}^{\frac{1}{2}}\cdots t_{j_r}^{\frac{1}{2}}$$

$$\left(\prod_{k\in F^+(p)} f_k^+\right)\left(\prod_{k\in F^-(p)} f_k^-\right)$$

$$F^+(p)=\{2,5\}$$

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$$t_1\tfrac{1}{2}t_2^{\frac{1}{2}}t_1^{\frac{1}{2}}\quad X^{\varepsilon_1+\varepsilon_2}t_2^{\frac{1}{2}}t_1^{\frac{1}{2}}t_2^{\frac{1}{2}}t_1^{\frac{1}{2}}\quad f_2^+f_5^+$$

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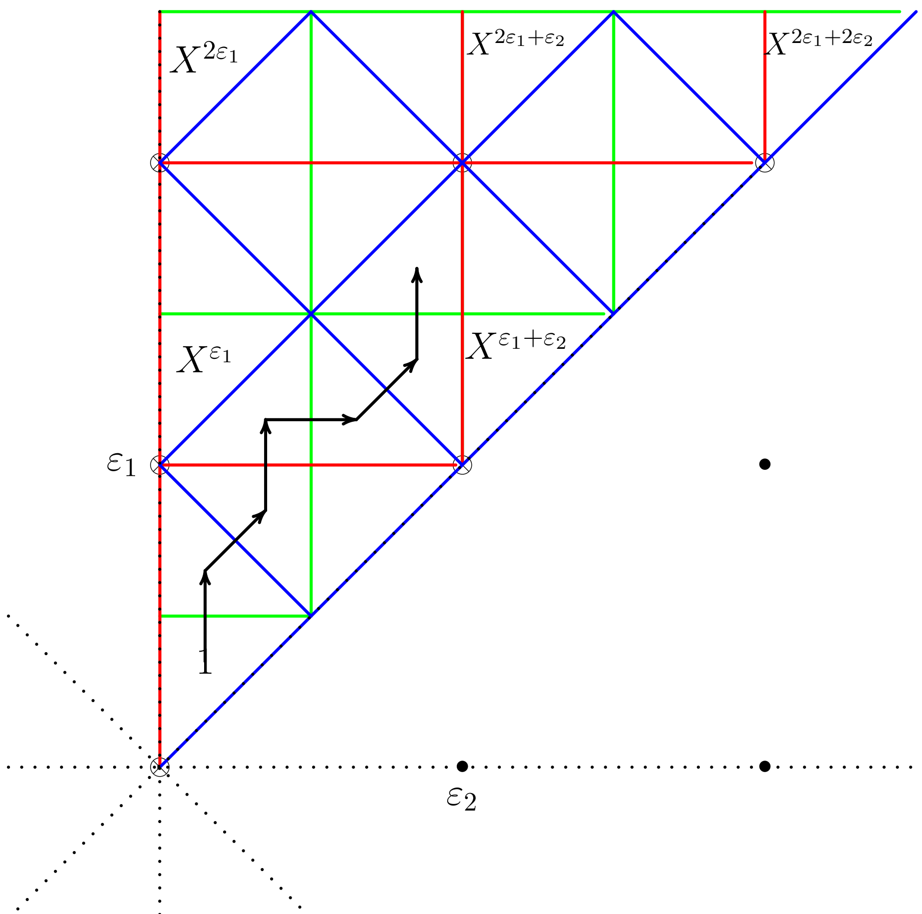
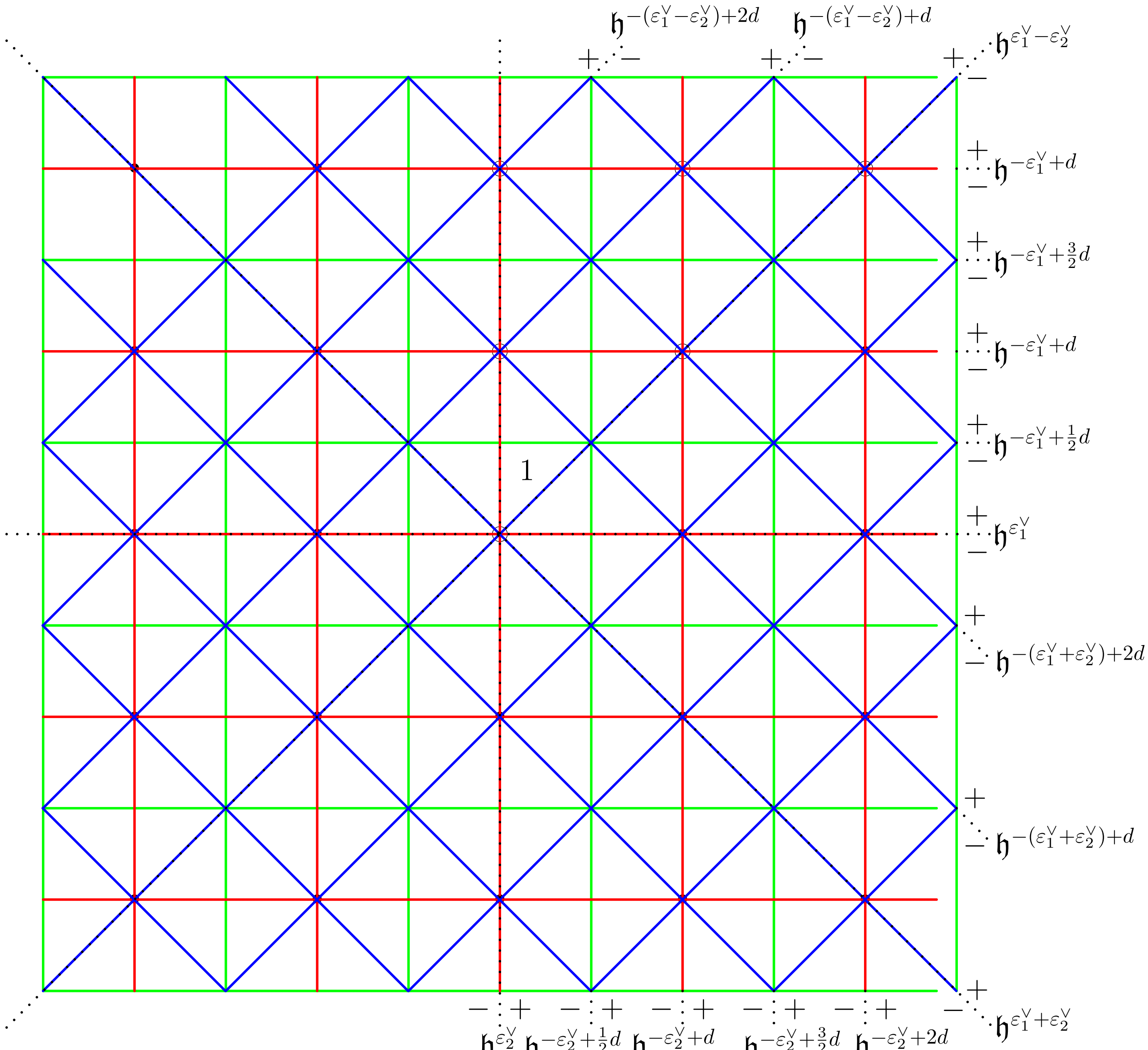
Approximately,

$$f_k^+ = \frac{t^{-\frac{1}{2}}(1-t) + t^{-\frac{1}{2}}(1-t) q^j Y^{-\beta_k^\vee}}{1 - q^{2j} Y^{2\beta_k^\vee}} \qquad \text{and} \qquad Y^{\varepsilon_i} = t_0^{\frac{1}{2}} t_2^{\frac{1}{2}} t_1^{n-i}$$

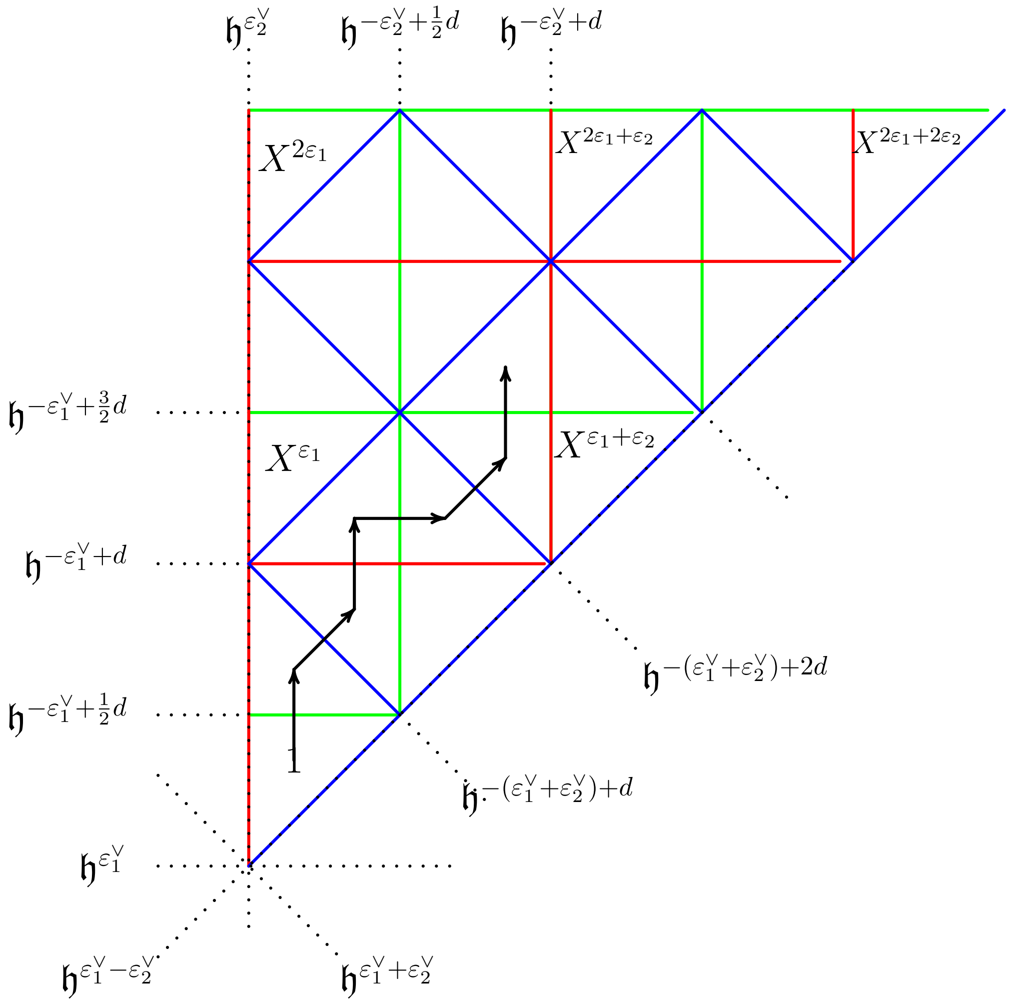
The original path p_λ, before folding



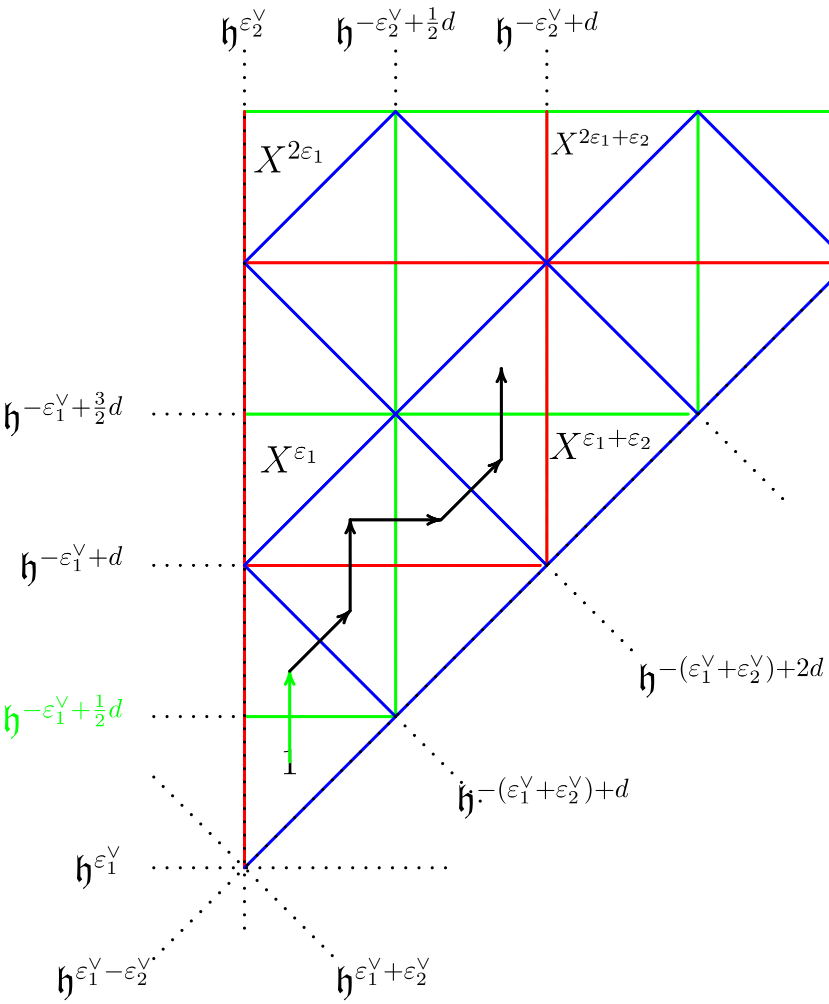


The affine Weyl group

The values f_k^+ and f_k^-, where $Y^{\varepsilon_i} = t_0^{\frac{1}{2}} t_2^{\frac{1}{2}} t_1^{n-i}$



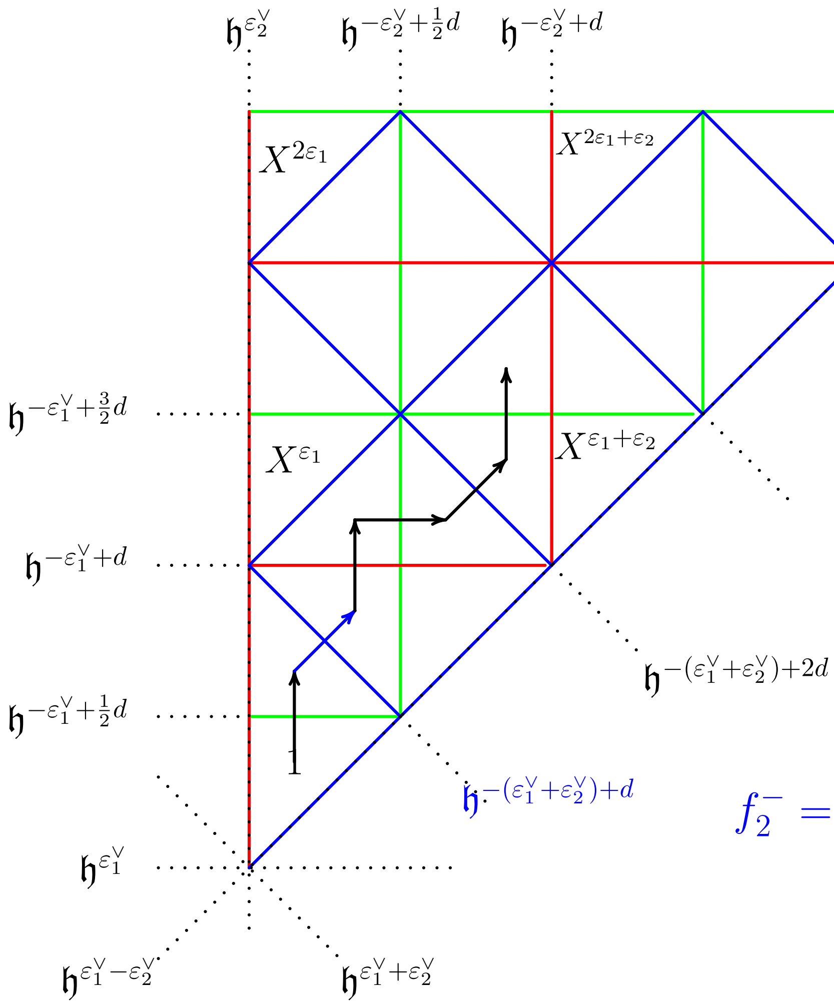
The values f_k^+ and f_k^-, where $Y^{\varepsilon_i} = t_0^{\frac{1}{2}} t_2^{\frac{1}{2}} t_1^{n-i}$



$$f_1^+ = \frac{u_0^{-\frac{1}{2}}(1-u_0) + u_2^{-\frac{1}{2}}(1-u_2)\, q^{\frac{1}{2}}\, Y^{\varepsilon_1^\vee}}{1 - q\, Y^{2\varepsilon_1^\vee}}$$

$$f_1^- = \frac{\left(u_0^{-\frac{1}{2}}(1-u_0) + u_2^{-\frac{1}{2}}(1-u_2)\, q^{-\frac{1}{2}}\, Y^{-\varepsilon_1^\vee}\right)\, q\, Y^{2\varepsilon_1^\vee}}{1 - q\, Y^{2\varepsilon_1^\vee}}$$

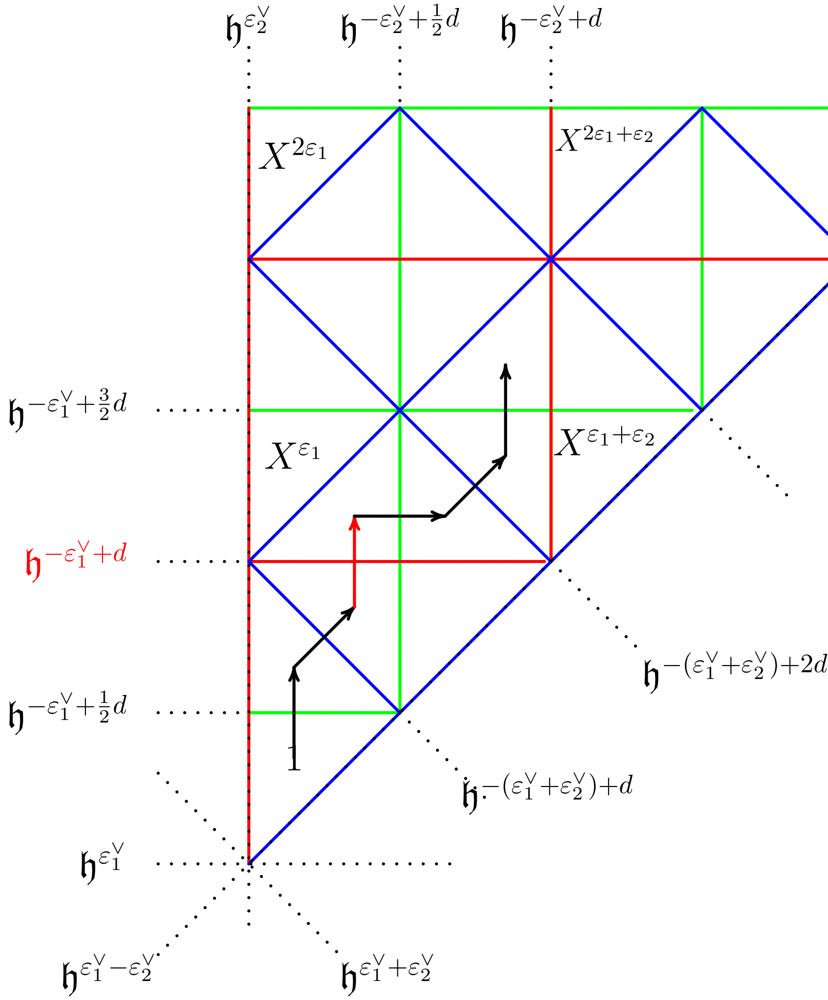
The values f_k^+ and f_k^-, where $Y^{\varepsilon_i} = t_0^{\frac{1}{2}} t_2^{\frac{1}{2}} t_1^{n-i}$



$$f_2^+ = \frac{t^{-\frac{1}{2}}(1-t) + t^{-\frac{1}{2}}(1-t)\, q\, Y^{\varepsilon_1^\vee+\varepsilon_2^\vee}}{1 - q^2\, Y^{2(\varepsilon_1^\vee+\varepsilon_2^\vee)}}$$

$$f_2^- = \frac{\left(t^{-\frac{1}{2}}(1-t) + t^{-\frac{1}{2}}(1-t)\, q^{-1}\, Y^{-(\varepsilon_1^\vee+\varepsilon_2^\vee)}\right)\, q^2\, Y^{2(\varepsilon_1^\vee+\varepsilon_2^\vee)}}{1 - q^2\, Y^{2(\varepsilon_1^\vee+\varepsilon_2^\vee)}}$$

The values f_k^+ and f_k^-, where $Y^{\varepsilon_i} = t_0^{\frac{1}{2}} t_2^{\frac{1}{2}} t_1^{n-i}$



$$f_3^+ = \frac{t_0^{-\frac{1}{2}}(1-t_0) + t_2^{-\frac{1}{2}}(1-t_2)\, q\, Y^{\varepsilon_1^\vee}}{1 - q^2\, Y^{2\varepsilon_1^\vee}}$$

$$f_3^- = \frac{\left(t_0^{-\frac{1}{2}}(1-t_0) + t_2^{-\frac{1}{2}}(1-t_2)\, q^{-1}\, Y^{-\varepsilon_1^\vee}\right)\, q^2\, Y^{2\varepsilon_1^\vee}}{1 - q^2\, Y^{2\varepsilon_1^\vee}}$$

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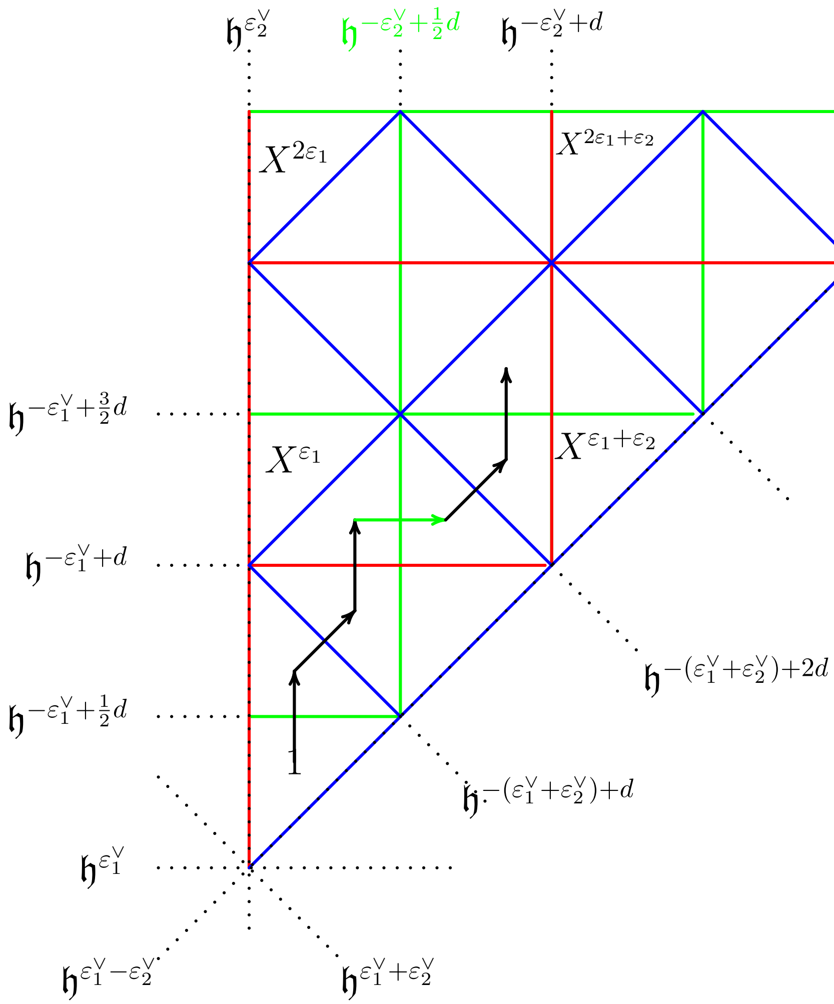

$$f_4^+ = \frac{u_0^{-\frac{1}{2}}(1-u_0) + u_2^{-\frac{1}{2}}(1-u_2)\, q^{\frac{1}{2}}\, Y^{\varepsilon_2^\vee}}{1 - q\, Y^{2\varepsilon_2^\vee}}$$

$$f_4^- = \frac{\left(u_0^{-\frac{1}{2}}(1-u_0) + u_2^{-\frac{1}{2}}(1-u_2)\, q^{-\frac{1}{2}}\, Y^{-\varepsilon_2^\vee}\right)\, q\, Y^{2\varepsilon_2^\vee}}{1 - q\, Y^{2\varepsilon_2^\vee}}$$

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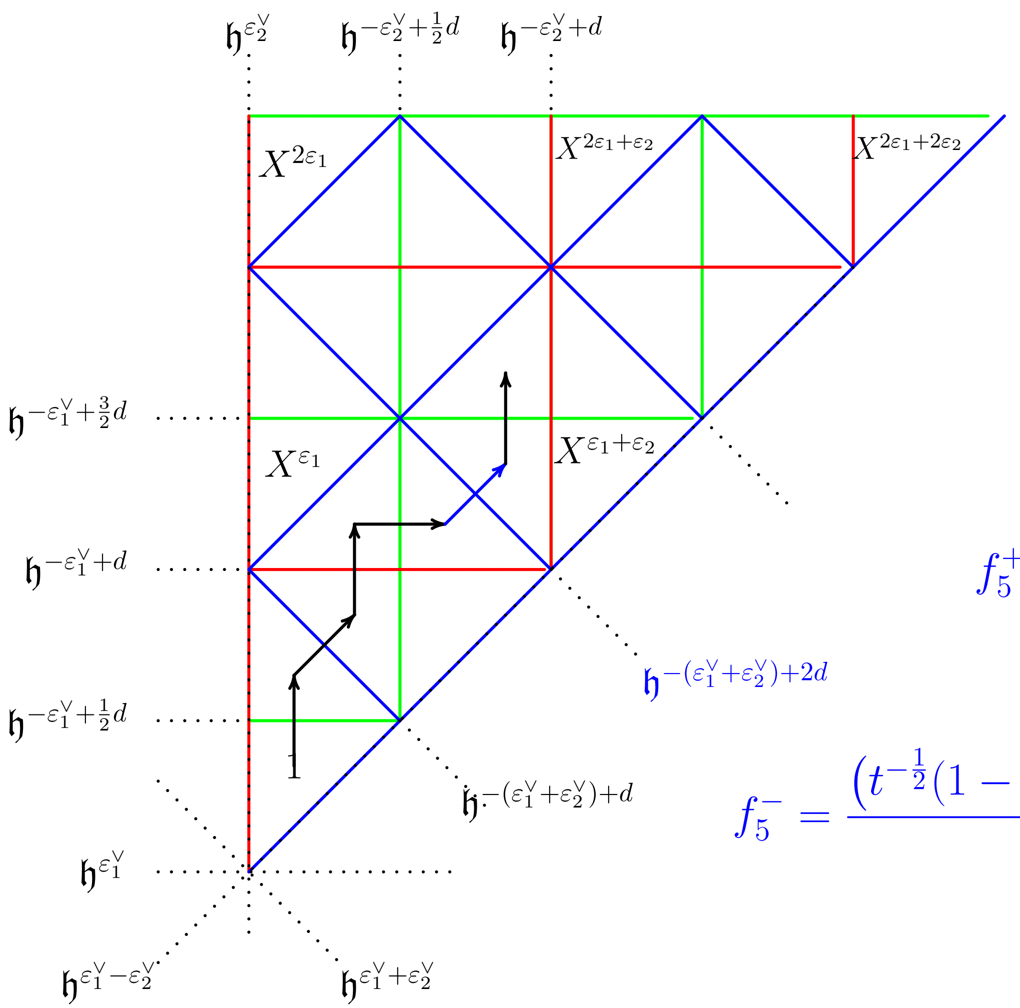

$$f_5^+ = \frac{t^{-\frac{1}{2}}(1-t) + t^{-\frac{1}{2}}(1-t)\,q^2\,Y^{\varepsilon_1^\vee+\varepsilon_2^\vee}}{1-q^4\,Y^{2(\varepsilon_1^\vee+\varepsilon_2^\vee)}}$$

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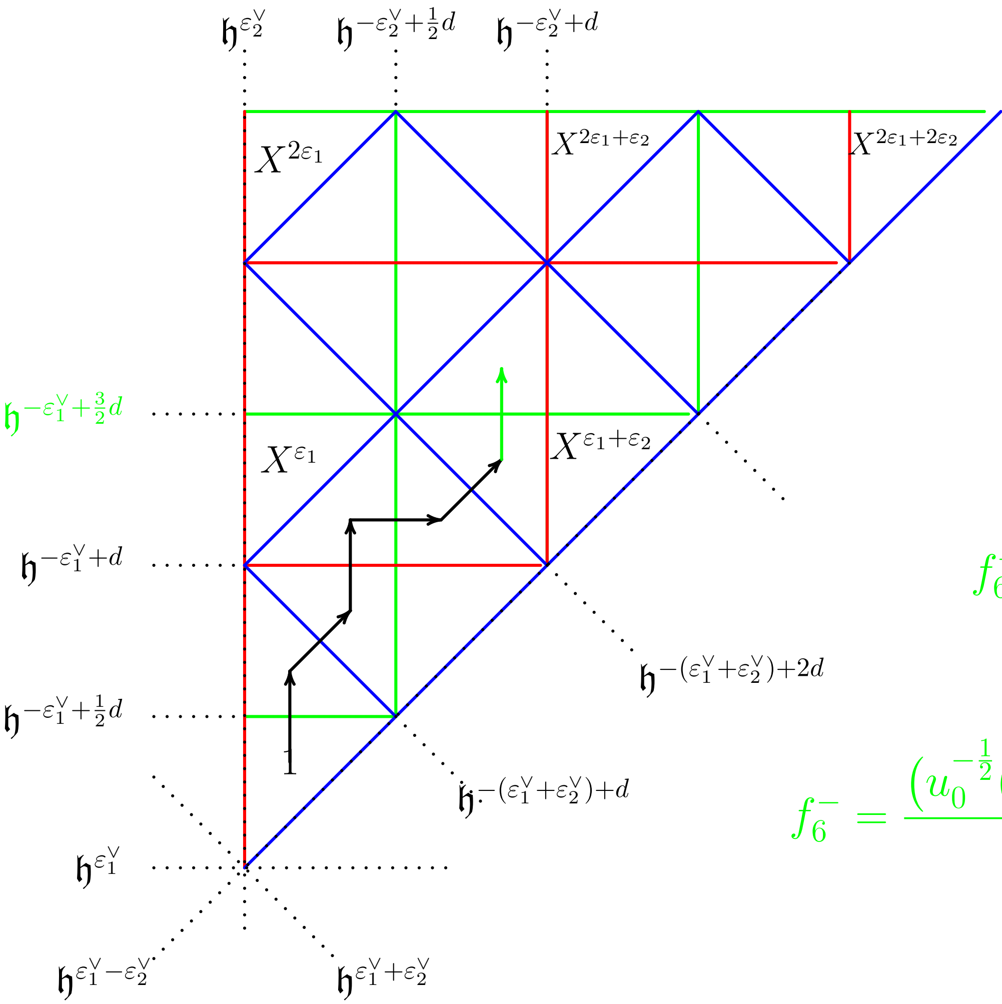

$$f_6^+ = \frac{u_0^{-\frac{1}{2}}(1-u_0) + u_2^{-\frac{1}{2}}(1-u_2)\, q^{\frac{3}{2}}\, Y^{\varepsilon_1^\vee}}{1 - q^3\, Y^{2\varepsilon_1^\vee}}$$

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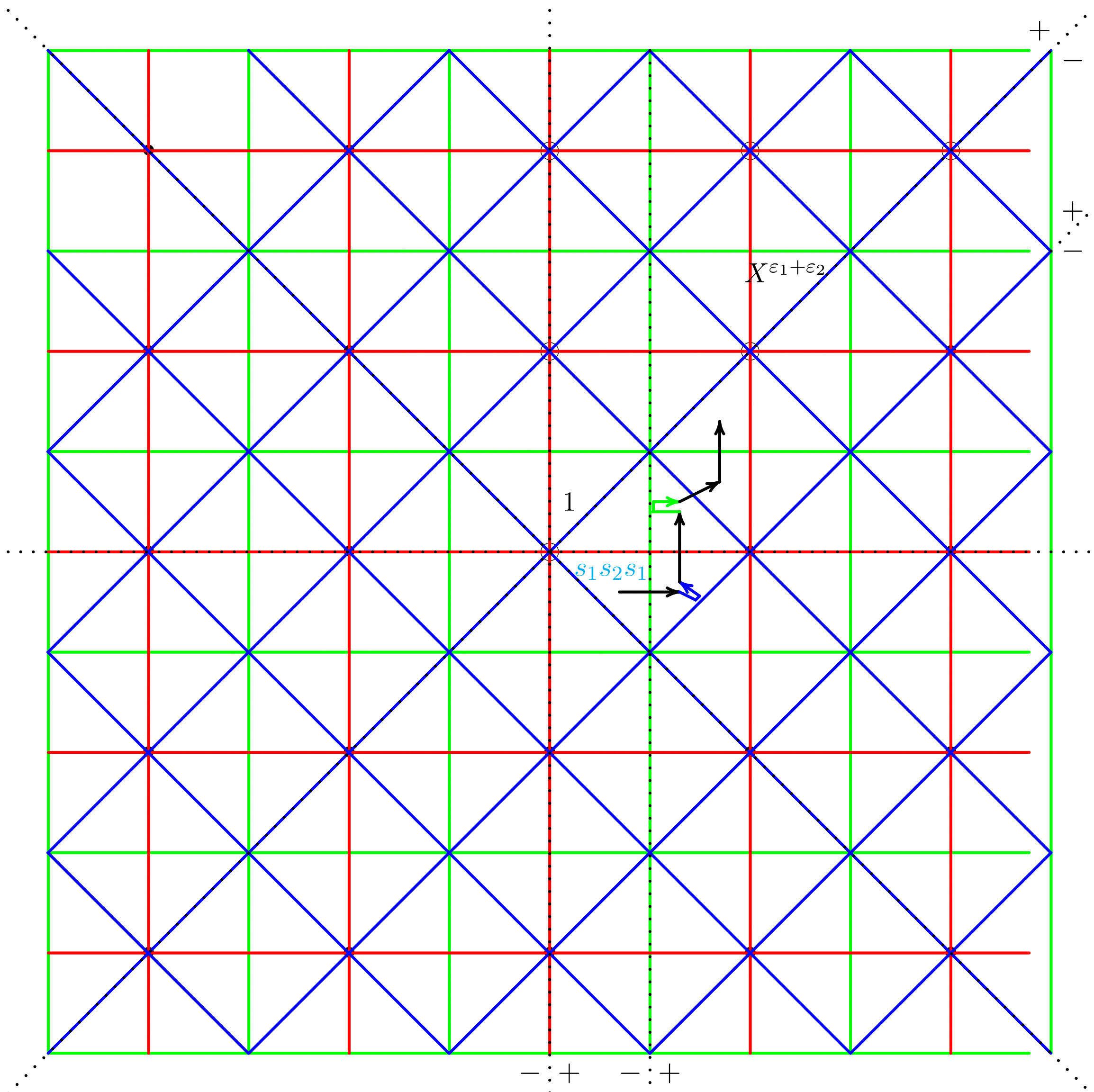

p begins at $s_{i_1}\dots s_{i_\ell}$

and ends at $X^{\mathrm{wt}(p)}s_{j_1}\cdots s_{j_r}$

$$t_{i_1}^{\frac{1}{2}}\cdots t_{i_\ell}^{\frac{1}{2}}\quad X^{\mathrm{wt}(p)}\ t_{j_1}^{\frac{1}{2}}\cdots t_{j_r}^{\frac{1}{2}}$$

$$\left(\prod_{k\in F^+(p)} f_k^+\right)\left(\prod_{k\in F^-(p)} f_k^-\right)$$

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Theorem (Ram-Yip)

Let $\lambda \in P^+$ (i.e. λ is a partition). Let p_λ be a minimal length path to the λ-octagon.

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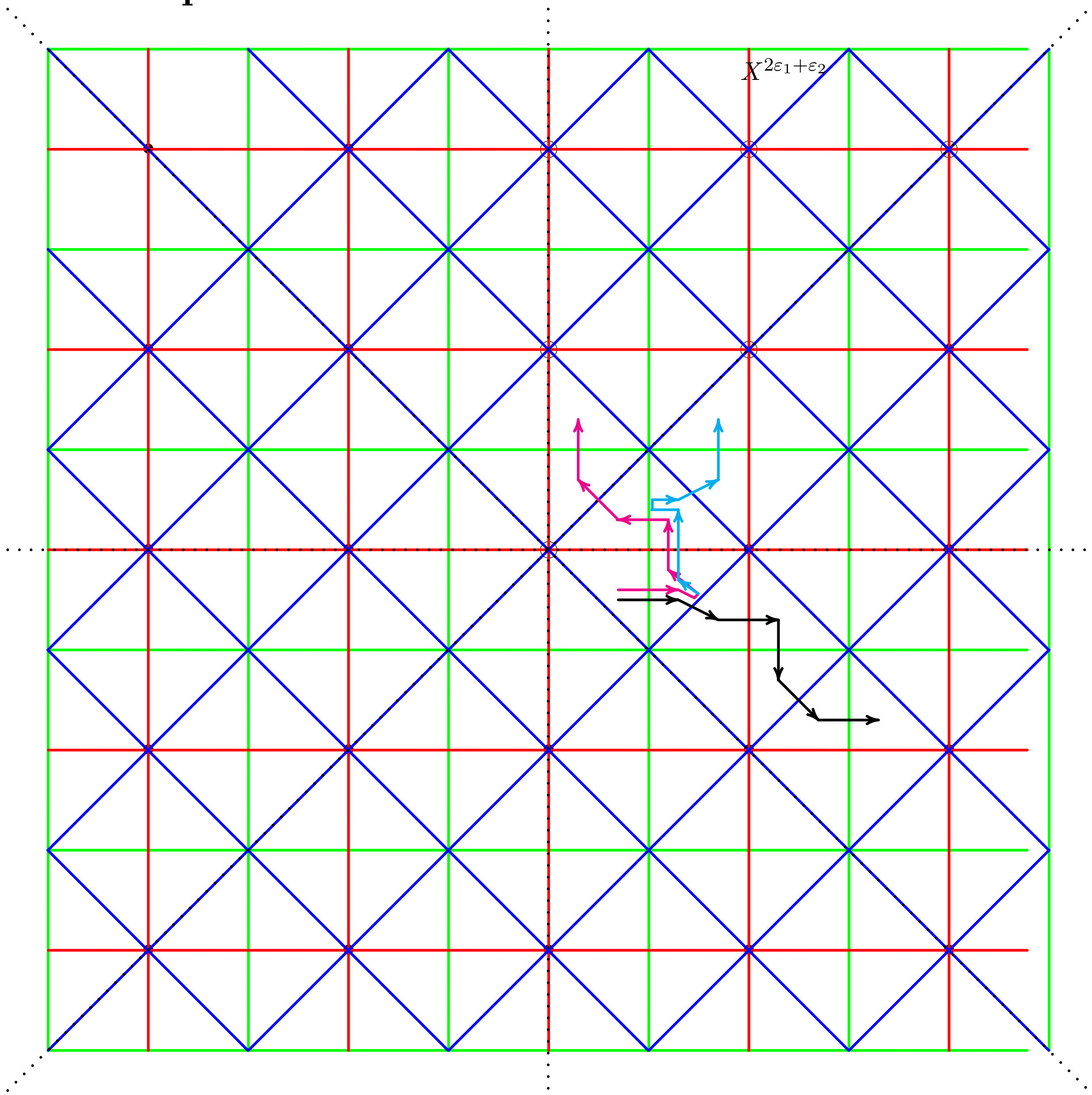
The point:



The Macdonald polynomial P_λ is given by a (weighted) sum over folded paths

Macdonald polynomials

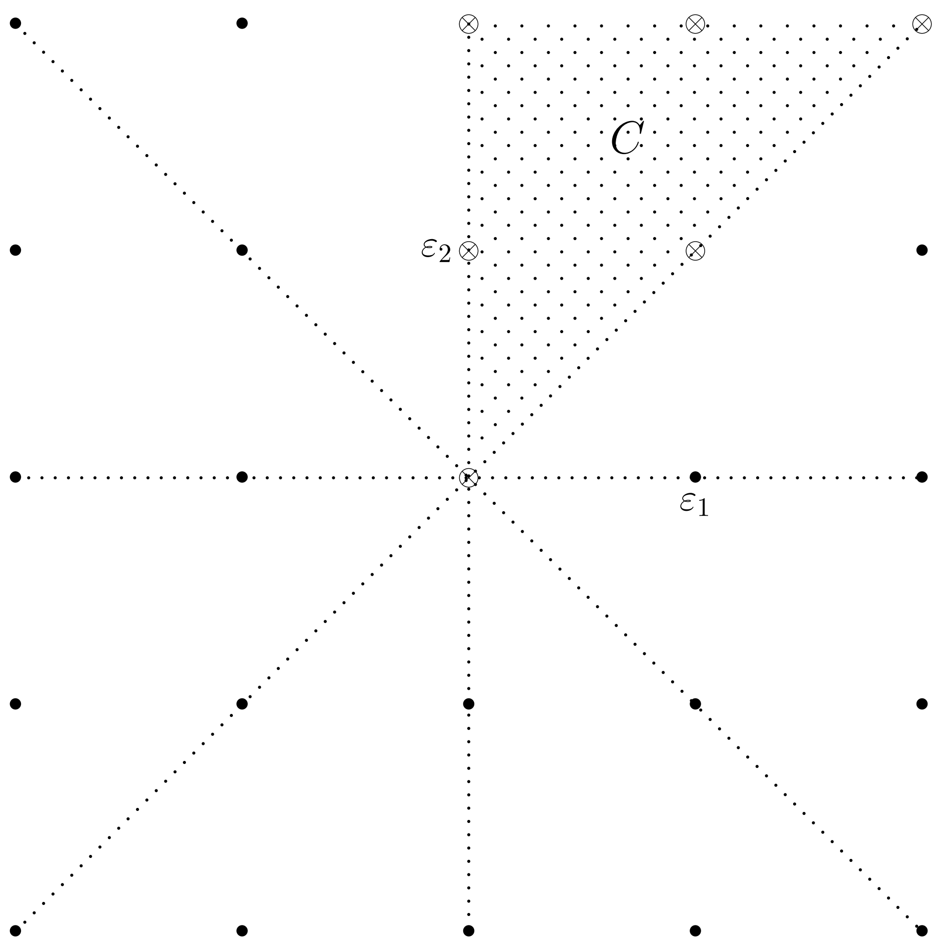
WHY?

Representation Theory: Let G be a compact Lie group.

Theorem (Weyl)

The irreducible G-modules $L(\lambda)$ are indexed by $\lambda \in P^+$.

Partitions λ are elements of P^+



Theorem (Weyl)

The irreducible G_0-modules $L(\lambda)$

are indexd by $\lambda \in P^+$

$P^+ = \mathfrak{h}^*_{\mathbb{Z}} \cap \overline{C}$

Representation Theory: Let G be a compact Lie group.

Theorem (Weyl)

The irreducible G-modules $L(\lambda)$ are indexed by $\lambda \in P^+$.

Nora would say:

$$\mathrm{Ell}_G(\mathrm{pt}) \longrightarrow K_G(\mathrm{pt}) \qquad \text{and}$$

$$K_G(\mathrm{pt}) = \mathrm{Rep}(G) \quad \text{has basis} \quad \{[L(\lambda)] \mid \lambda \in P^+\}.$$

Representation Theory: Let G be a compact Lie group.

Theorem (Weyl)

The irreducible G-modules $L(\lambda)$ are indexed by $\lambda \in P^+$.

Borel-Weil-Bott say:

$$L(\lambda) = H^0(G/B, \mathcal{L}_\lambda), \qquad \text{where}$$

G/B is the *flag variety* and $\mathcal{L}_\lambda$ is the line bundle $\begin{array}{c} G \times_B \mathbb{C}_\lambda \\ \downarrow \\ G/B \end{array}$

Representation Theory: Let G be a compact Lie group.

Theorem (Weyl)

The irreducible G-modules $L(\lambda)$ are indexed by $\lambda \in P^+$.

Borel-Weil-Bott say: $L(\lambda) = H^0(G/B, \mathcal{L}_\lambda)$

Alex says: $H^0(X(N), \mathcal{L}^{\otimes k})$ is the space of *modular forms*

of weight k and level $\Gamma(N)$, where

$E_{/X(N)}$ is a universal (generalised) elliptic curve and

$\mathcal{L}$ is the *Hodge bundle* $\pi_* \Omega^1_{E_{/X(N)}}$.

Representation Theory: Let G be a compact Lie group.

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Representation Theory: Let G be a compact Lie group.

$\mathcal{L}_\lambda$ is the line bundle $\begin{array}{c} G \times_B \mathbb{C}_\lambda \\ \downarrow \\ G/B \end{array}$

Craig says:

$$\begin{array}{ccc} \mathrm{Rep}(G_0) & \longrightarrow & K_{G_0}(\mathrm{pt})^\wedge \\ & & EG_0 \times_{G_0} L \\ L & \longmapsto & \downarrow \\ & & BG_0 \end{array}$$

Representation Theory: Let G be a compact Lie group.

Theorem (Weyl)

The irreducible G-modules $L(\lambda)$ are indexed by $\lambda \in P^+$.

Macdonald says: Put $X^\mu = 1$, $t_i^{\frac{1}{2}} = 0$ and $q^{\frac{1}{2}} = 0$ in P_λ.

Then P_λ specialises to $\dim(L(\lambda))$.

More generally,

$$P_\lambda\Big|_{t=q=0} \quad = \quad \mathrm{char}(L(\lambda))$$

Representation Theory: Let G be a compact Lie group.

Theorem (Weyl)

The irreducible G-modules $L(\lambda)$ are indexed by $\lambda \in P^+$.

$\mathrm{char}(L(\lambda))$ is a specialisation of P_λ

So $\mathrm{char}(L(\lambda))$ **is a weighted sum of folded paths.**

(previously known formula of Littelmann)

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

$$
\begin{array}{lcl}
G = G_0(\mathbb{C}((t))) & & \\
\cup\!| & & \\
K = G_0(\mathbb{C}[[t]]) & \xrightarrow{t=0} & G_0(\mathbb{C}) \\
\cup\!| & & \cup\!| \\
I = \Phi^{-1}(B) & \longrightarrow & B \quad = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}
\end{array}
$$

Harmonic Analysis: Let G_0 be a compact Lie group.

Harmonic Analysis: Let G_0 be a simple algebraic group.

Harmonic Analysis: Let G_0 be a reductive algebraic group.

Harmonic Analysis: Let G_0 be a complex reductive algebraic group.

Harmonic Analysis: Let $G_0 = GL_n$

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

$$G = G_0(\mathbb{C}((t)))$$

$$\cup|$$

$$K = G_0(\mathbb{C}[[t]])$$

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

$$G = G_0(\mathbb{Q}_p)$$

$$\cup|$$

$$K = G_0(\mathbb{Z}_p)$$

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

$$G = G_0(\mathbb{F}_q((t)))$$

$$\cup|$$

$$K = G_0(\mathbb{F}_q[[t]])$$

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

$$G = G_0(\mathbb{C}((t)))$$

$$\cup|$$

$$K = G_0(\mathbb{C}[[t]]) \xrightarrow{t=0} G_0(\mathbb{C})$$

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\cup| \\
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\end{array}
$$

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

$$G = G_0(\mathbb{F}_q((t)))$$

$$\cup|$$

$$K = G_0(\mathbb{F}_q[[t]]) \xrightarrow{t=0} G_0(\mathbb{F}_q)$$

Harmonic Analysis: Let $G_0 = GL_n$ be a compact Lie group.

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G = G_0(\mathbb{C}((t))) & & & \\
\cup\!| & & & \\
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\end{array}
$$

$G_0(\mathbb{C}((t))) = \mathrm{Map}(S^1, G_0)$

is the *loop group*

G/K is the *loop Grassmannian*

G/I is the *affine flag variety*

Harmonic Analysis: $G = G_0(\mathbb{C}((t)))$

The main problem in **Geometric Langlands**

is to *really* understand the isomorphism

$$\mathrm{Rep}(G_0^\vee) \quad \xrightarrow{\sim} \quad C(K\backslash G/K)$$

$$C(K\backslash G/K) = \{\text{functions } f\colon G \to \mathbb{C} \text{ such that } f(k_1 g k_2) = f(g).\}$$

Let $\chi_{Kt_\lambda K}$ be the characteristic function of $Kt_\lambda K$.

Harmonic Analysis: $G = G_0(\mathbb{C}((t)))$

Macdonald says

$$\begin{array}{ccc} \mathrm{Rep}(G_0^\vee) & \xrightarrow{\sim} & C(K\backslash G/K) \\ P_\lambda\big|_{q=0} & \longmapsto & \chi_{Kt_\lambda K} \end{array}$$

$\chi_{Kt_\lambda K}$ is the characteristic function of $Kt_\lambda K$.

$$G = \bigsqcup_{\lambda \in P^+} Kt_\lambda K, \qquad \text{where} \quad t_\lambda = \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix}$$

Harmonic Analysis: $G = G_0(\mathbb{C}((t)))$

$$\begin{array}{ccc} \mathrm{Rep}(G_0^\vee) & \xrightarrow{\sim} & C(K\backslash G/K) \\ P_\lambda\big|_{q=0} & \longmapsto & \chi_{Kt_\lambda K} \end{array}$$

The spherical function is a specialisation of P_λ

So the spherical function is a weighted sum of folded paths

(previously known formula of Schwer)

Perhaps

Does P_λ have something to do with $G_0(\mathbb{C}((s))((t)))$??

Is $G_0(\mathbb{C}((s))((t))) \;=\; \text{Map}(\text{torus}, G_0)$ the *elliptic group*?