Affine braids, Markov traces and the category $\mathcal{O}$

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1. Introduction

This paper provides a unified approach to results on representations of affine Hecke algebras, cyclotomic Hecke algebras, affine BMW algebras, cyclotomic BMW algebras, Markov traces, Jacobi-Trudi type identities, dual pairs [Ze], and link invariants [Tu2]. The key observation in the genesis of this paper was that the technical tools used to obtain the results in Orellana [Or] and Suzuki [Su], two a priori unrelated papers, are really the same. Here we develop this method and explain how to apply it to obtain results similar to those in [Or] and [Su] in more general settings. Some specific new results which are obtained are the following:

(a) A generalization of the results on Markov traces obtained by Orellana [Or] to centralizer algebras coming from quantum groups of all Lie types.

(b) A generalization of the results of Suzuki [Su] to show that Kazhdan-Lusztig polynomials of all finite Weyl groups occur as decomposition numbers in the representation theory of affine braid groups of type A,

(c) A generalization of the functors used by Zelevinsky [Ze] to representations of affine braid groups of type A,

(d) We define the affine BMW algebra (Birman-Murakami-Wenzl) and show that it has a representation theory analogous to that of affine Hecke algebras. In particular there are "standard modules" for these algebras which have composition series where multiplicities of the factors are given by Kazhdan-Lusztig polynomials for Weyl groups of types A,B,and C.

(e) We generalize the results of Leduc and Ram [LR] to affine centralizer algebras.

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Let $U_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group associated to a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. If M is a (possibly infinite dimensional) $U_h\mathfrak{g}$-module in the category $\mathcal{O}$ and V is a finite dimensional $U_h\mathfrak{g}$-module then we show that the affine braid group $\tilde{\mathcal{B}}_k$ acts on the $U_h\mathfrak{g}$-module $M \otimes V^{\otimes k}$. Fix V and define

$$F_\lambda(M) = \begin{pmatrix} \text{the vector space of highest weight vectors} \\ \text{of weight } \lambda \text{ in } M \otimes V^{\otimes k}. \end{pmatrix}.$$

Then F_λ is a functor from $U_h\mathfrak{g}$ modules in category $\mathcal{O}$ to finite dimensional modules for the affine braid group $\tilde{\mathcal{B}}_k$ which takes

(1) finite dimensional $U_h\mathfrak{g}$ modules to "calibrated" $\tilde{\mathcal{B}}_k$ modules,

(2) Verma modules to "standard" modules, and

(3) under appropriate conditions, irreducible $U_h\mathfrak{g}$ modules to irreducible $\tilde{\mathcal{B}}_k$ modules.

Applying the functor F_λ to a Jantzen filtration of Verma modules of $U_h\mathfrak{g}$ provides a "Jantzen filtration" of the standard modules of $\tilde{\mathcal{B}}_k$ and shows that the irreducible $\tilde{\mathcal{B}}_k$ modules appear in a composition series of the standard module with multiplicities given by the Kazhdan-Lusztig polynomials of the Weyl group of $\mathfrak{g}$. Though $\tilde{\mathcal{B}}_k$ is always the affine braid group of type A, the Weyl group of $\mathfrak{g}$ is *not usually* of type A.

Applying the functor F_λ to the BGG resolution of an irreducible highest weight module provides a BGG resolution for the corresponding $\tilde{\mathcal{B}}_k$-modules and a corresponding "Jacobi-Trudi" identity for the characters of $\tilde{\mathcal{B}}_k$ modules. Once again, it is interesting to note that, though $\tilde{\mathcal{B}}_k$ is the affine braid group of type A, it is the Weyl group of a different type which appears in this Jacobi-Trudi identity.

Using the general formulation for constructing Markov traces on braid groups, given for example in [Tu1], we obtain a Markov trace on the affine braid group $\tilde{\mathcal{B}}_k$ for every choice of $\mathfrak{g}$ and $U_h\mathfrak{g}$ modules M and V.

(a) If $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $M = L(0)$ and $V = L(\omega_1)$ this gives the Markov trace on the Hecke algebra studied in [Jo1] and [Wz].

(b) If $\mathfrak{g} = \mathfrak{sl}_2$, $M = L(0)$ and $V = L(\omega_1)$ this gives the Markov trace on the Temperley-Lieb algebra used by Jones [Jo2].

(c) If $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $M = L(k\omega_\ell)$ with k and ℓ large and n very large, and $V = L(\omega_1)$ this gives the Markov traces on the Hecke algebra of type B studied by [GL], [Lb], [Ic] and [Or].

(d) If $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $M = L(\lambda)$, where λ is "large", and $V = L(\omega_1)$ this gives the Markov traces on the cyclotomic Hecke algebras introduced by Lambropoulou [Lb] and studied in [GIM].

(e) If $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{g} = \mathfrak{sp}_{2n}$, $M = L(0)$ and $V = L(\omega_1)$ this gives the Markov traces used to construct Kauffman polynomials.

For general $\mathfrak{g}$, general V, and $M = L(0)$, this mechanism gives the traces necessary to compute the Reshetikhin-Turaev link invariants [RT]. In some sense, this paper is a study of the representation theory behind the generalization of the Reshetikhin-Turaev method given in [Tu2].

In the final section of this paper we describe precisely the combinatorics of the representations $F_\lambda(M)$ in the cases when $\mathfrak{g}$ is type A_n, B_n, C_n or D_n and V is the fundamental representation. In these cases the representations can be constructed with partitions, standard tableaux, up-down tableaux, multisegments and the combinatorics of Young diagrams. In particular, in type A, the functor F_λ naturally constructs the standard modules and irreducible modules of affine Hecke algebras of type A in terms of multisegments (a classification originally obtained by Zelevinsky [Ze2] by different methods). We then specify explicitly the correspondence between the decomposition numbers of the affine Hecke algebra and Kazhdan-Lusztig polynomials for the symmetric group.

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2. Preliminaries on quantum groups

Let $U_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. Let us fix some notations. In particular, fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \mathfrak{n}^+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \qquad \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+,$$

and let W be the Weyl group of $\mathfrak{g}$. Let $\langle , \rangle$ be the usual inner product on $\mathfrak{h}^*$ so that, if α is a root, the corresponding reflection s_α in W is given by

$$s_\alpha\lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \qquad \text{where} \quad \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}. \qquad \text{The element} \quad \rho = \frac{1}{2}\sum_{\alpha>0} \alpha$$

is often viewed as an element of $\mathfrak{h}$ by using the form $\langle , \rangle$ to identify $\mathfrak{h}$ and $\mathfrak{h}^*$. We shall use the conventions for quantum groups as in [Dr] and [LR] so that

$$q = e^{h/2}, \qquad \mathfrak{h} \subseteq U_h\mathfrak{g}, \qquad \text{and} \quad U_h\mathfrak{g} \cong U\mathfrak{g}[[h]], \quad \text{as algebras.}$$

The quantum group has a triangular decomposition corresponding to that of $\mathfrak{g}$,

$$U_h\mathfrak{g} = U_h\mathfrak{n}^- \otimes U_h\mathfrak{h} \otimes U_h\mathfrak{n}^+ \qquad \text{and} \qquad U_h\mathfrak{b}^+ = U_h\mathfrak{h} \otimes U_h\mathfrak{n}^+.$$

The category $\mathcal{O}$

If M is a $U_h\mathfrak{g}$ module and $\lambda \in \mathfrak{h}^*$ the λ weight space of M is

$$M_\lambda = \{m \in M \mid am = \lambda(a)m, \text{ for all } a \in \mathfrak{h}\}.$$

The category $\mathcal{O}$ is the category of $U_h\mathfrak{g}$ modules M such that

(a) $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$,

(b) For all $m \in M$, $\dim(U_h\mathfrak{n}^+m)$ is finite,

(c) M is finitely generated as a $U_h\mathfrak{g}$ module.

For $\mu \in \mathfrak{h}^*$ let

$M(\mu)$ be the Verma module of highest weight μ, and let

$L(\mu)$ be the irreducible module of highest weight μ.

The irreducible module $L(\mu)$ is the quotient of $M(\mu)$ by a maximal proper submodule and $M(\mu) = U_h\mathfrak{g} \otimes_{U_h\mathfrak{b}^+} \mathbb{C}v_\mu^+$ where $\mathbb{C}v_\mu^+$ is the one dimensional $U_h\mathfrak{b}^+$ module spanned by a vector v_μ^+ such that $av_\mu^+ = \mu(a)v_\mu^+$ for $a \in \mathfrak{h}$ and $U_h\mathfrak{n}^+v_\mu^+ = 0$. Every module $M \in \mathcal{O}$ has a finite composition series with factors $L(\mu)$, $\mu \in \mathfrak{h}^*$. Each of the sets

$$\{[L(\lambda)] \mid \lambda \in \mathfrak{h}^*\} \qquad \text{and} \qquad \{[M(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$$

(where $[M]$ denotes the isomorphism class of the module M) are bases of the Grothendieck group of the category $\mathcal{O}$.

If M is a $U_h\mathfrak{g}$ module generated by a highest weight vector of weight λ (i.e., a vector v^+ such that $av^+ = \lambda(a)v^+$ for $a \in \mathfrak{h}$ and $U_h\mathfrak{n}^+v^+ = 0$) then any element of the center $Z(U_h\mathfrak{g})$ acts on M by a constant,

$$zm = \chi^\lambda(z)m, \quad \text{for } z \in Z(U_h\mathfrak{g}),\ m \in M,\ \chi^\lambda(z) \in \mathbb{C}.$$

For each $U_h\mathfrak{g}$ module $M \in \mathcal{O}$ let

$$M^{[\lambda]} = \bigoplus_{\nu \in Q} M^{[\lambda]}_{\lambda+\nu}, \qquad \text{where} \quad Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i,$$

$\alpha_1, \ldots, \alpha_n$ are the simple roots and

$$M^{[\lambda]}_{\lambda+\nu} = \{m \in M_{\lambda+\nu} \mid \text{there is } k \in \mathbb{Z}_{>0} \text{ such that } (z - \chi^\lambda(z))^k m = 0 \text{ for all } z \in Z(U_h\mathfrak{g})\}.$$

Then

$$M = \bigoplus_{\lambda} M^{[\lambda]},$$

where the sum is over all *integrally dominant* weights $\lambda \in \mathfrak{h}^*$ i.e., $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda+\rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in R^+$. To summarize, there is a decomposition of the category $\mathcal{O}$,

$$\mathcal{O} = \bigoplus_{\lambda} \mathcal{O}^{[\lambda]}, \tag{2.1}$$

where the sum is over all integrally dominant weights $\lambda \in \mathfrak{h}^*$ and $\mathcal{O}^{[\lambda]}$ is the full subcategory of modules $M \in \mathcal{O}$ such that $M = M^{[\lambda]}$. The Grothendieck group of the category $\mathcal{O}^{[\lambda]}$ has bases

$$\{[L(\mu)] \mid \mu \in W^\lambda \circ \lambda\} \qquad \text{and} \qquad \{[M(\mu)] \mid \mu \in W^\lambda \circ \lambda\}. \tag{2.2}$$

where the *integral Weyl group* corresponding to λ is

$$W^\lambda = \langle s_\alpha \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \rangle \qquad \text{and} \qquad w \circ \lambda = w(\lambda + \rho) - \rho, \qquad w \in W, \lambda \in \mathfrak{h}^*. \tag{2.3}$$

defines the *dot action* of W on $\mathfrak{h}^*$.

Jantzen filtrations

Following the notations for the quantum group used in [LR, §2], let $\mathfrak{h}$, $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ be the standard generators of the quantum group $U_h\mathfrak{g}$ which satisfy the quantum Serre relations. The *Cartan involution* $\theta\colon U_h\mathfrak{g} \to U_h\mathfrak{g}$ is the algebra anti-involution defined by

$$\theta(X_i) = Y_i, \qquad \theta(Y_i) = X_i, \qquad \text{and} \qquad \theta(a) = a, \quad \text{for } a \in \mathfrak{h}. \tag{2.4}$$

The Cartan involution θ is a coalgebra homomomorphism. A *contravariant form* on a $U_h\mathfrak{g}$ module M is a symmetric bilinear form $\langle , \rangle\colon M \times M \to \mathbb{C}$ such that

$$\langle um_1, m_2 \rangle = \langle m_1, \theta(u)m_2 \rangle, u \in U_h\mathfrak{g}, \quad m_1, m_2 \in M.$$

Fix $\lambda \in \mathfrak{h}^*$ and $\delta \in \mathfrak{h}^*$ such that $\lambda + t\delta$ is integrally dominant for all small positive real numbers t. Consider t as an indeterminate and consider the Verma module

$$M(\lambda + t\delta) = U_h\mathfrak{g}[t] \otimes_{U_h\mathfrak{b}^+[t]} \mathbb{C}_{\lambda+t\delta}$$

as the module for $U_h\mathfrak{g}[t] = \mathbb{C}[t] \otimes_{\mathbb{C}} U_h\mathfrak{g}$ generated by a vector v^+ such that $av^+ = (\lambda + t\delta)(a)v^+$ for $a \in \mathfrak{h}$ and $U_h\mathfrak{n}^+[t]v^+ = 0$. There is a unique contravariant form $\langle , \rangle_t \colon M(\lambda + t\delta) \times M(\lambda + t\delta) \to \mathbb{C}[t]$ such that $\langle v^+, v^+ \rangle_t = 1$. Define

$$M(\lambda + t\delta)(j) = \{m \in M(\lambda + t\delta) \mid \langle m, n \rangle_t \in t^j\mathbb{C}[t] \text{ for all } n \in M(\lambda + t\delta)\}.$$

The "specialization of $M(\lambda + t\delta)(j)$ at $t = 0$" is

$$M(\lambda)^{(j)} = \text{image of } M(\lambda + t\delta)(j) \text{ in } M(\lambda + t\delta) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/t\mathbb{C}[t]$$

and the *Jantzen filtration* of $M(\lambda)$ is

$$M(\lambda) = M(\lambda)^{(0)} \supseteq M(\lambda)^{(1)} \supseteq \cdots . \tag{2.5}$$

By [Jz, Theorem 5.3], the Jantzen filtration is a filtration of $M(\lambda)$ by $U_h\mathfrak{g}$ modules, the module $M(\lambda)^{(1)}$ is a maximal proper submodule of $M(\lambda)$ and each quotient $M(\lambda)^{(i)}/M(\lambda)^{(i+1)}$ has a nondegenerate contravariant form. It is known [Bb] that the Jantzen filtration does not depend on the choice of δ. It is a deep theorem [BB] that the quotients $M(\lambda)^{(i)}/M(\lambda)^{(i+1)}$ are semisimple and that if $w \in W^\mu$ and $y \in W^\mu$ are maximal length in their cosets $wW_{\mu+\rho}$ and $yW_{\mu+\rho}$, respectively, then the Kazhdan-Lusztig polynomial for W^μ is

$$\sum_{j \geq 0} [M(w \circ \mu)^{(j)}/M(w \circ \mu)^{(j+1)} : L(y \circ \mu)] \mathtt{v}^{\frac{1}{2}(\ell(y) - \ell(w) - j)} = P_{wy}(\mathtt{v}), \tag{2.6}$$

where ℓ is the length function on W^μ and $[M(w \circ \mu)^{(j)}/M(w \circ \mu)^{(j+1)} : L(y \circ \mu)]$ is the multiplicity of the simple module $L(y \circ \mu)$ in the jth factor of the Jantzen filtration of $M(\lambda)$.

The BGG resolution

Not all simple modules $L(\lambda)$ in the category $\mathcal{O}$ have a BGG resolution. The general form of the BGG resolution given by Gabber and Joseph [GJ] is as follows.

Let $\mu \in \mathfrak{h}^*$ be such that $-(\mu + \rho)$ is dominant and regular and let W_J^μ be a parabolic subgroup of the integral Weyl group W^μ. Let w_0 be the longest element of W_J^μ and fix $\nu = w_0 \circ \mu$. Define a resolution

$$0 \longrightarrow C_{\ell(w_0)} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow L(\nu) \longrightarrow 0 \tag{2.7}$$

of the simple module $L(\nu)$ by Verma modules by setting

$$C_j = \bigoplus_{\ell(w)=j} M(w \circ \nu),$$

where the sum is over all $w \in W_J^\mu$ of length j, and defining the map

$$d_j \colon C_j \to C_{j-1}, \quad \text{by the matrix} \quad (d_j)_{v,w} = \begin{cases} \varepsilon_{v,w}\iota_{v,w}, & \text{if } v \to w, \\ 0, & \text{otherwise,} \end{cases},$$

where $v \to w$ means that there is a (not necessarily simple) root α such that $w = s_\alpha v$ and $\ell(w) = \ell(v) - 1$, the maps $\iota_{v,w}$ are fixed choices of inclusions

$$\iota_{v,w}\colon M(v \circ \nu) \hookrightarrow M(w \circ \nu), \qquad \text{and} \qquad \varepsilon_{v,w} = \pm 1,$$

are fixed choices of signs such that

$$\varepsilon_{u,v}\varepsilon_{v,w} = -\varepsilon_{u,v'}\varepsilon_{v',w} \qquad \text{if} \quad u \to v \to w, \quad u \to v' \to w \quad \text{and} \quad v \neq v'.$$

Gabber and Joseph [GJ] prove that the sequence (2.7) is exact in this general setting. See [BGG] and [Dx,7.8.14] for the original form of the BGG resolution. From the exactness of (2.7) it follows that if $-(\mu + \rho)$ is dominant and regular then, in the Grothendieck group of the category $\mathcal{O}$,

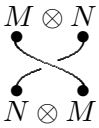
$$[L(\nu)] = \sum_{w \in W_J^\mu} (-1)^{\ell(w)} [M(w \circ \nu)], \tag{2.8}$$

where $\nu = w_0 \circ \mu$ and w_0 is the longest element of W_J^μ.

$\check{R}_{MN}$ matrices and the quantum Casimir C_M

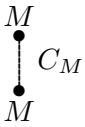
Let $U_h\mathfrak{g}$ be the Drinfeld-Jimbo quantum group corresponding to a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. There is an invertible element $\mathcal{R} = \sum a_i \otimes b_i$ in (a suitable completion of) $U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ such that, for any two $U_h\mathfrak{g}$ modules M and N, the map

$$\begin{array}{rccc} \check{R}_{MN}\colon & M \otimes N & \longrightarrow & N \otimes M \\ & m \otimes n & \longmapsto & \sum b_i n \otimes a_i m \end{array}$$

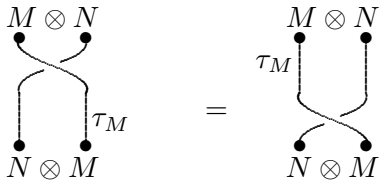


is a $U_h\mathfrak{g}$ module isomorphism. There is also a *quantum Casimir element* $e^{-h\rho}u$ in the center of $U_h\mathfrak{g}$ and, for a $U_h\mathfrak{g}$ module M we define

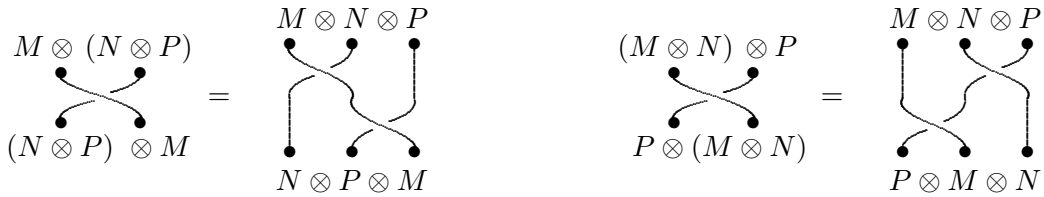
$$\begin{array}{rccc} C_M\colon & M & \longrightarrow & M \\ & m & \longmapsto & (e^{-h\rho}u)m \end{array}$$



In order to be consistent with the graphical calculus these operators should be written *on the right*. The elements $\mathcal{R}$ and $e^{-h\rho}u$ satisfy relations (see [LR, (2.1-2.12)]) which imply that, for $U_h\mathfrak{g}$ modules M, N, P and a $U_h\mathfrak{g}$ module isomorphism $\tau_M\colon M \to M$,



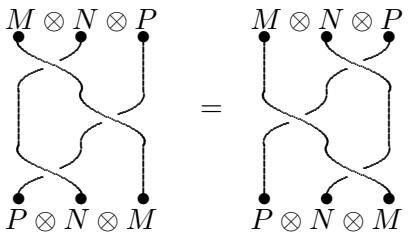
$$\check{R}_{MN}(\mathrm{id}_N \otimes \tau_M) = (\tau_M \otimes \mathrm{id}_N)\check{R}_{MN}, \tag{2.9}$$



$$\check{R}_{M,N\otimes P} = (\check{R}_{MN} \otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP}) \qquad \check{R}_{M\otimes N,P} = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \mathrm{id}_N), \tag{2.10}$$

$$C_{M\otimes N} = (\check{R}_{MN}\check{R}_{NM})^{-1}(C_M \otimes C_N). \tag{2.11}$$

The relations (2.9) and (2.10) together imply the braid relation



(2.12)

$$(\check{R}_{MN}\otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP})(\check{R}_{NP}\otimes \mathrm{id}_M) = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP}\otimes \mathrm{id}_N)(\mathrm{id}_P \otimes \check{R}_{MN}),$$

If M is a $U_h\mathfrak{g}$ module generated by a highest weight vector v^+ of weight λ then, by [Dr, Prop. 3.2],

$$C_M = q^{-\langle\lambda,\lambda+2\rho\rangle}\mathrm{id}_M. \tag{2.13}$$

Note that $\langle\lambda,\lambda+2\rho\rangle = \langle\lambda+\rho,\lambda+\rho\rangle - \langle\rho,\rho\rangle$ are the eigenvalues of the classical Casimir operator [Dx, 7.8.5]. If M is a finite dimensional $U_h\mathfrak{g}$ module then M is a direct sum of the irreducible modules $L(\lambda)$, $\lambda \in P^+$, and

$$C_M = \bigoplus_{\lambda\in P^+} q^{-\langle\lambda,\lambda+2\rho\rangle}P_\lambda,$$

where $P_\lambda\colon M \to M$ is the projection onto $M^{[\lambda]}$ in M. From the relation (2.11) it follows that if $M = L(\mu)$, $N = L(\nu)$ are finite dimensional irreducible $U_h\mathfrak{g}$ modules then $\check{R}_{MN}\check{R}_{NM}$ acts on the λ isotypic component $L(\lambda)^{\oplus c^\lambda_{\mu\nu}}$ of the decomposition

$$L(\mu)\otimes L(\nu) = \bigoplus_\lambda L(\lambda)^{\oplus c^\lambda_{\mu\nu}} \qquad \text{by the constant} \qquad q^{\langle\lambda,\lambda+2\rho\rangle-\langle\mu,\mu+2\rho\rangle-\langle\nu,\nu+2\rho\rangle}. \tag{2.14}$$

Suppose that M and N are $U_h\mathfrak{g}$ modules with contravariant forms $\langle,\rangle_M$ and $\langle,\rangle_N$, respectively. Since the Cartan involution is a coalgebra homomorphism the form on $M\otimes N$ defined by

$$\langle m_1\otimes n_1, m_2\otimes n_2\rangle = \langle m_1,m_2\rangle_M\langle n_1,n_2\rangle_N, \qquad \text{for } m_1,m_2\in M,\ n_1,n_2\in N, \tag{2.15}$$

is also contravariant. If θ is the Cartan involution defined in (2.4) then a formula of Drinfeld [Dr, Prop. 4.2] states

$$(\theta\otimes\theta)(\mathcal{R}) = \sum_i b_i\otimes a_i,$$

from which it follows that

$$\begin{aligned}
\langle (m_1\otimes n_1)\check{R}_{MN}, n_2\otimes m_2\rangle &= \sum_i \langle (b_i\otimes a_i)(n_1\otimes m_1), n_2\otimes m_2\rangle \\
&= \sum_i \langle n_1\otimes m_1, (\theta(b_i)\otimes\theta(a_i))(n_2\otimes m_2)\rangle \\
&= \sum_i \langle n_1\otimes m_1, (a_i\otimes b_i)(n_2\otimes m_2)\rangle \\
&= \sum_i \langle m_1\otimes n_1, b_i m_2\otimes a_i n_2\rangle.
\end{aligned}$$

Thus

$$\langle (m_1\otimes n_1)\check{R}_{MN}, n_2\otimes m_2\rangle = \langle m_1\otimes n_1, (n_2\otimes m_2)\check{R}_{NM}\rangle. \tag{2.16}$$

3. Affine braid group representations and the functors F_λ

There are three common ways of depicting affine braids [Cr], [GL], [Jo3]:

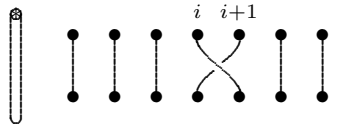
(a) As braids in a (slightly thickened) cylinder,

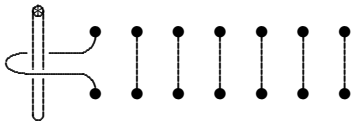
(b) As braids in a (slightly thickened) annulus,

(c) As braids with a flagpole.

See Figure 1. The multiplication is by placing one cylinder on top of another, placing one annulus inside another, or placing one flagpole braid on top of another. These are equivalent formulations: an annulus can be made into a cylinder by turning up the edges, and a cylindrical braid can be made into a flagpole braid by putting a flagpole down the middle of the cylinder and pushing the pole over to the left so that the strings begin and end to its right.

The *affine braid group* is the group $\tilde{\mathcal{B}}_k$ formed by the affine braids with k strands. The affine braid group $\tilde{\mathcal{B}}_k$ can be presented by generators $T_1, T_2, \dots, T_{k-1}$ and X^{ε_1}

$T_i =$ 

and $X^{\varepsilon_1} =$  (3.1)

with relations

(3.2a) $T_iT_j = T_jT_i$, if $|i-j|>1$,

(3.2b) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $1 \le i \le k-2$,

(3.2c) $X^{\varepsilon_1}T_1X^{\varepsilon_1}T_1 = T_1X^{\varepsilon_1}T_1X^{\varepsilon_1}$,

(3.2d) $X^{\varepsilon_1}T_i = T_iX^{\varepsilon_1}$, for $2 \le i \le k-1$.

Define

$$X^{\varepsilon_i} = T_{i-1}T_{i-2}\cdots T_2T_1X^{\varepsilon_1}T_1T_2\cdots T_{i-1}, \qquad 1 \le i \le k. \tag{3.3}$$

By drawing pictures of the corresponding affine braids it is easy to check that the X^{ε_i} all commute with each other and so $X = \langle X^{\varepsilon_i} \mid 1 \le i \le k\rangle$ is an abelian subgroup of $\tilde{\mathcal{B}}_k$. Let $L \cong \mathbb{Z}^k$ be the free abelian group generated by $\varepsilon_1, \dots, \varepsilon_k$. Then

$$L = \{\lambda_1\varepsilon_1 + \cdots + \lambda_k\varepsilon_k \mid \lambda_i \in \mathbb{Z}\} \qquad \text{and} \qquad X = \{X^\lambda \mid \lambda \in L\}, \tag{3.4}$$

where $X^\lambda = (X^{\varepsilon_1})^{\lambda_1}(X^{\varepsilon_2})^{\lambda_2}\cdots(X^{\varepsilon_k})^{\lambda_k}$, for $\lambda \in L$.

The $\tilde{\mathcal{B}}_k$ module $M \otimes V^{\otimes k}$

Let $U_h\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. Let M be a $U_h\mathfrak{g}$-module in the category $\mathcal{O}$ and let V be a finite dimensional $U_h\mathfrak{g}$ module. Define $\check{R}_i$, $1 \le i \le k-1$, and $\check{R}_0^2$ in $\mathrm{End}_{U_h\mathfrak{g}}(M \otimes V^{\otimes k})$ by

$$\check{R}_i = \mathrm{id}_M \otimes \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}_V^{\otimes(k-i-1)} \qquad \text{and} \qquad \check{R}_0^2 = (\check{R}_{MV}\check{R}_{VM}) \otimes \mathrm{id}_V^{\otimes(k-1)}.$$

The following proposition is well known (see [Su2, Prop. B.2], [LR, Prop. 2.19], or [Re]).

Proposition 3.5. *The map defined by*

$$\begin{array}{rccll} \Phi: & \tilde{\mathcal{B}}_k & \longrightarrow & End_{U_h\mathfrak{g}}(M \otimes V^{\otimes k}) & \\ & T_i & \longmapsto & \check{R}_i, & 1 \le i \le k-1, \\ & X^{\varepsilon_1} & \longmapsto & \check{R}_0^2, & \end{array}$$

makes $M \otimes V^{\otimes k}$ into a right $\tilde{\mathcal{B}}_k$ module.

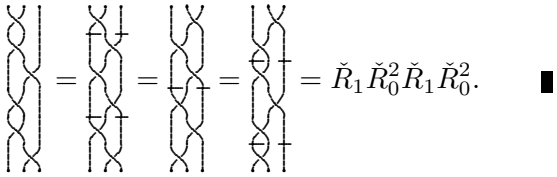
Proof. It is necessary to show that

(a) $\check{R}_i \check{R}_j = \check{R}_j \check{R}_i$, if $|i-j|>1$,
(b) $\check{R}_0^2 \check{R}_i = \check{R}_i \check{R}_0^2$, $i>2$,
(c) $\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1}$, $1 \le i \le k-2$,
(d) $\check{R}_0^2 \check{R}_1 \check{R}_0^2 \check{R}_1 = \check{R}_1 \check{R}_0^2 \check{R}_1 \check{R}_0^2$.

The relations (a) and (b) follow immediately from the definitions of $\check{R}_i$ and $\check{R}_0^2$ and (c) is a particular case of the braid relation (2.12). The relation (d) is also a consequence of the braid relation:

$$\begin{aligned}
\check{R}_0^2 \check{R}_1 \check{R}_0^2 \check{R}_1 &= (\check{R}_{MV}\check{R}_{VM} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{VV})(\check{R}_{MV}\check{R}_{VM} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{VV}) \\
&= (\check{R}_{MV} \otimes \mathrm{id})\, \underbrace{(\mathrm{id} \otimes \check{R}_{MV})(\check{R}_{VV} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{VM})}(\check{R}_{VM} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{VV}) \\
&= \underbrace{(\mathrm{id} \otimes \check{R}_{VV})(\check{R}_{MV} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{MV})}\, \underbrace{(\check{R}_{VV} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{VM})(\check{R}_{VM} \otimes \mathrm{id})} \\
&= (\mathrm{id} \otimes \check{R}_{VV})(\check{R}_{MV} \overbrace{\check{R}_{VM} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{VV})(\check{R}_{MV}}\, \check{R}_{VM} \otimes \mathrm{id}) \\
&= \check{R}_1 \check{R}_0^2 \check{R}_1 \check{R}_0^2,
\end{aligned}$$

or equivalently,

$\check{R}_0^2 \check{R}_1 \check{R}_0^2 \check{R}_1 =$  $= \check{R}_1 \check{R}_0^2 \check{R}_1 \check{R}_0^2.$ ∎

A $\tilde{\mathcal{B}}_k$ module N is *calibrated* if the abelian group X defined in (3.4) acts semisimply on N, i.e. if N has a basis of simultaneous eigenvectors for the action of $X^{\varepsilon_1}, \dots, X^{\varepsilon_k}$.

Proposition 3.6. *If M and V are finite dimensional $U_h\mathfrak{g}$ modules then the $\tilde{\mathcal{B}}_k$ module $M \otimes V^{\otimes k}$ defined in Proposition 3.5 is calibrated.*

Proof. Let P^+ be the set of dominant integral weights. Since M and V are finite dimensional the $U_h\mathfrak{g}$-module $M \otimes V^{\otimes i}$ is semisimple for every $1 \le i \le k$ and

$$M \otimes V^{\otimes i} = \bigoplus_{\lambda \in P^+} (M \otimes V^{\otimes i})^{[\lambda]} \cong \bigoplus_{\lambda \in P^+} L(\lambda)^{\oplus m_\lambda},$$

where $m_\lambda \in \mathbb{Z}_{\ge 0}$ and $(M \otimes V^{\otimes i})^{[\lambda]} \cong L(\lambda)^{\oplus m_\lambda}$. Given a basis of $M \otimes V^{\otimes(i-1)}$ which respects the decomposition $M \otimes V^{\otimes(i-1)} = \bigoplus_\mu (M \otimes V^{\otimes(i-1)})^{[\mu]}$ one can construct a basis of $M \otimes V^{\otimes i}$ which respects the decomposition

$$M \otimes V^{\otimes i} = (M \otimes V^{\otimes(i-1)}) \otimes V = \bigoplus_{\lambda,\mu,\nu} ((M \otimes V^{\otimes(i-1)})^{[\mu]} \otimes V^{[\nu]})^{[\lambda]}.$$

Since $((M \otimes V^{\otimes(i-1)})^{[\mu]} \otimes V^{[\nu]})^{[\lambda]} \subseteq (M \otimes V^{\otimes i})^{[\lambda]}$ this new basis respects the decomposition $M \otimes V^{\otimes i} = \bigoplus_\lambda (M \otimes V^{\otimes i})^{[\lambda]}$. This procedure produces, inductively, a basis B of $M \otimes V^{\otimes k}$ which respects the decompositions

$$M \otimes V^{\otimes k} = (M \otimes V^{\otimes i}) \otimes V^{\otimes(k-i)} = \bigoplus_\lambda (M \otimes V^{\otimes i})^{[\lambda]} \otimes V^{\otimes(k-i)},$$

for all $0 \le i \le k$. The central element $e^{-h\rho}u$ in $U_h\mathfrak{g}$ acts on $(M\otimes V^{\otimes i})^{[\lambda]}$ by the constant $q^{-\langle\lambda,\lambda+2\rho\rangle}$. From (2.10), (2.11) and (2.14) it follows that X^{ε_i} acts on $M\otimes V^{\otimes k}$ by

$$\begin{aligned}\check{R}_{i-1}\cdots\check{R}_1\check{R}_0^2\check{R}_1\cdots\check{R}_{i-1} &= \check{R}_{M\otimes V^{\otimes(i-1)},V}\check{R}_{V,M\otimes V^{\otimes(i-1)}}\otimes \mathrm{id}_V^{\otimes(k-i)}\\ &= (C_{M\otimes V^{\otimes(i-1)}}\otimes C_V)C^{-1}_{M\otimes V^{\otimes i}}\otimes \mathrm{id}_V^{\otimes(k-i)}\\ &= \sum_{\lambda,\mu,\nu} q^{\langle\lambda,\lambda+2\rho\rangle-\langle\mu,\mu+2\rho\rangle-\langle\nu,\nu+2\rho\rangle}P^\lambda_{\mu\nu}\otimes \mathrm{id}_V^{\otimes(k-i)}\end{aligned}$$

where $P^\lambda_{\mu\nu}\colon M\otimes \mathrm{id}_V^{\otimes i}\to M\otimes \mathrm{id}_V^{\otimes i}$ is the projection onto $((M\otimes V^{\otimes(i-1)})^{[\mu]}\otimes V^{[\nu]})^{[\lambda]}$. Thus X^{ε_i} acts diagonally on the basis B. ∎

Define an anti-involution on $\tilde{\mathcal{B}}_k$ by

$$\tilde{\theta}(T_i) = T_i \qquad \text{and} \qquad \tilde{\theta}(X^\lambda) = X^\lambda,$$

for $1\le i\le k-1$ and $\lambda\in L$. A *contravariant form* on a $\tilde{\mathcal{B}}_k$ module N is a symmetric bilinear form $\langle,\rangle\colon N\times N\to\mathbb{C}$ such that

$$\langle bn_1, n_2\rangle = \langle n_1, \tilde{\theta}(b)n_2\rangle \qquad \text{for } n_1,n_2\in N,\ b\in\tilde{\mathcal{B}}_k.$$

Suppose M is a $U_h\mathfrak{g}$-module in the category $\mathcal{O}$ and V is a finite dimensional $U_h\mathfrak{g}$ module. Let $\langle,\rangle_M$ and $\langle,\rangle_V$ be $U_h\mathfrak{g}$-contravariant forms on M and V respectively. By (2.16),

$$\langle(v_1\otimes v_2)\check{R}_{VV}, v_1'\otimes v_2'\rangle = \langle v_1\otimes v_2, (v_1'\otimes v_2')\check{R}_{VV}\rangle$$

for $v_1,v_2,v_1',v_2'\in V$, and

$$\langle(m\otimes v)\check{R}_{MV}\check{R}_{VM}, m'\otimes v'\rangle = \langle(m\otimes v)\check{R}_{MV},(m'\otimes v')\check{R}_{MV}\rangle = \langle m\otimes v,(m'\otimes v')\check{R}_{MV}\check{R}_{VM}\rangle$$

for $m,m'\in M$, $v,v'\in V$. Thus it follows that the form $\langle,\rangle$ on $M\otimes V^{\otimes k}$ given by

$$\langle m\otimes v_1\otimes\cdots v_k, m'\otimes v_1'\otimes\cdots v_k'\rangle = \langle m,m'\rangle_M\langle v_1,v_1'\rangle_V\langle v_2,v_2'\rangle_V\cdots\langle v_k,v_k'\rangle_V, \tag{3.7}$$

for $m,m'\in M$, $v_i,v_i'\in V$ is a $\tilde{\mathcal{B}}_k$ contravariant form on the $\tilde{\mathcal{B}}_k$ module $M\otimes V^{\otimes k}$.

The functor F_λ

Fix a finite dimensional $U_h\mathfrak{g}$ module V and an integrally dominant weight λ in $\mathfrak{h}^*$. Let $\tilde{\mathcal{O}}_k$ be the category of finite dimensional $\tilde{\mathcal{B}}_k$ modules and define a functor

$$\begin{array}{rccc} F_\lambda\colon & \mathcal{O} & \longrightarrow & \tilde{\mathcal{O}}_k\\ & M & \longmapsto & \mathrm{Hom}_{U_h\mathfrak{g}}(M(\lambda), M\otimes V^{\otimes k}).\end{array} \tag{3.8}$$

Since an element of $\mathrm{Hom}_{U_h\mathfrak{g}}(M(\lambda),M\otimes V^{\otimes k})$ is determined by the image of a generating highest weight vector of $M(\lambda)$, the space $F_\lambda(M)$ can be identified with the vector space of highest weight vectors of weight λ in $M\otimes V^{\otimes k}$. If λ is integrally dominant the highest possible weight of $(M\otimes V^{\otimes k})^{[\lambda]}$ is λ. Thus, viewing $F_\lambda(M)$ as the space of highest weight vectors of weight λ in $M\otimes V^{\otimes k}$,

$$\mathrm{Hom}_{U_h\mathfrak{g}}(M(\lambda),M\otimes V^{\otimes k}) \cong ((M\otimes V^{\otimes k})^{[\lambda]})_\lambda \cong \left(\frac{M\otimes V^{\otimes k}}{\mathfrak{n}^-(M\otimes V^{\otimes k})}\right)_\lambda$$

where we use the notation

$$\mathfrak{n}^-(M \otimes V^{\otimes k}) = \sum_i Y_i(M \otimes V^{\otimes k}), \tag{3.9}$$

where the Y_i are the Chevalley generators of $\mathfrak{n}^-$. In the case of $U\mathfrak{g}$-modules, the notation $\mathfrak{n}^-(M \otimes V^{\otimes k})$ is self explanatory—the notation in (3.9) is simply a way to define the same object for the quantum group $U_h\mathfrak{g}$.

The functor F_λ is the composition of two functors: the functor $\cdot \otimes V^{\otimes k}$ and the functor $\mathrm{Hom}_U(M(\lambda), \cdot)$. The first is exact since $V^{\otimes k}$ is finite dimensional and the second is exact because when λ is integrally dominant $M(\lambda)$ is projective, see [Jz, p. 72]. Thus

$$\text{if } \lambda \text{ is integrally dominant,} \quad \text{the functor } F_\lambda \text{ is exact.} \tag{3.10}$$

4. The $\tilde{\mathcal{B}}_k$ modules $\mathcal{M}^{\lambda/\mu}$ and $\mathcal{L}^{\lambda/\mu}$

Let λ be integrally dominant and let $\mu \in \mathfrak{h}^*$. Define $\tilde{\mathcal{B}}_k$ modules

$$\mathcal{M}^{\lambda/\mu} = F_\lambda(M(\mu)) \qquad \text{and} \qquad \mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu)). \tag{4.1}$$

The following lemma is the main tool for studying the structure of these $\tilde{\mathcal{B}}_k$ modules.

Lemma 4.2. *([Jz, Theorem 2.2], [Dx, Lemma 7.6.14]) Let E be a finite dimensional $U_h\mathfrak{g}$ module and let $\{e_i\}$ be a basis of E consisting of weight vectors ordered so that $i < j$ if $wt(e_i) < wt(e_j)$. Suppose M is a $U_h\mathfrak{g}$ module generated by a highest weight vector v_μ^+ of weight μ. Set*

$$M_i = \sum_{j \geq i} U_h\mathfrak{n}^-(v_\mu^+ \otimes e_j).$$

Then

(a) $M \otimes E = M_1 \supseteq M_2 \supseteq \cdots$ is a filtration of $U_h\mathfrak{g}$ modules such that M_i/M_{i+1} is 0 or is a highest weight module of highest weight $\mu + wt(e_i)$.

(b) If $M = M(\mu)$ then $M_i/M_{i+1} \cong M(\mu + wt(e_i))$.

The braid group $\mathcal{B}_k$ is the subgroup of $\tilde{\mathcal{B}}_k$ generated by $T_1, \ldots, T_{k-1}$. By restriction, both $\mathcal{M}^{\lambda/\mu}$ and $V^{\otimes k} = L(0) \otimes V^{\otimes k}$ are $\mathcal{B}_k$ modules.

There is a unique $U_h\mathfrak{g}$ contravariant form $\langle , \rangle_M$ on the Verma module $M(w \circ \mu)$ determined by $\langle v_{w\circ\mu}^+, v_{w\circ\mu}^+ \rangle_M = 1$ where $v_{w\circ\mu}^+$ is the generating highest weight vector of $M(w \circ \mu)$. As in (2.15), this form together with a nondegenerate $U_h\mathfrak{g}$ contravariant form $\langle , \rangle_V$ on V gives $U_h\mathfrak{g}$ contravariant forms $\langle , \rangle_{V^{\otimes k}}$ and $\langle , \rangle$ on $V^{\otimes k}$ and $M(w \circ \mu) \otimes V^{\otimes k}$, respectively.

With these notations at hand we use Lemma 4.2 to prove the fundamental facts about the $\tilde{\mathcal{B}}_k$ modules $\mathcal{M}^{\lambda/\mu}$ and $\mathcal{L}^{\lambda/\mu}$ defined in (4.1).

Proposition 4.3. *Let λ, μ be integrally dominant weights and $w \in W^\mu$.*

(a) As $\mathcal{B}_k$ modules, $\mathcal{M}^{\lambda/w\circ\mu} \cong (V^{\otimes k})_{\lambda - w\circ\mu}$

(b) $\mathcal{M}^{\lambda/w\circ\mu} \cong \mathcal{M}^{\lambda/y\circ\mu}$ if $W_{\lambda+\rho} w W_{\mu+\rho} = W_{\lambda+\rho} y W_{\mu+\rho}$.

(c) Use the same notation $\langle , \rangle$ for the $U_h\mathfrak{g}$ contravariant form $\langle , \rangle$ on $M(w \circ \mu) \otimes V^{\otimes k}$ and the $\tilde{\mathcal{B}}_k$ contravariant form on $\mathcal{M}^{\lambda/(w\circ\mu)}$ obtained by restriction of $\langle , \rangle$ to the subspace $(M(w \circ \mu) \otimes V^{\otimes k})_\lambda^{[\lambda]}$. Then

$$\mathcal{L}^{\lambda/w\circ\mu} \cong \frac{\mathcal{M}^{\lambda/(w\circ\mu)}}{\mathrm{rad}\langle , \rangle}.$$

(d) *Assume $w \in W^\mu$ is maximal length in the coset $wW_{\mu+\rho}$ in W^μ. If $\mathcal{L}^{\lambda/(w\circ\mu)} \neq 0$ then*
   (1) $\lambda - w\circ\mu$ is a weight of $V^{\otimes k}$,
   (2) w is maximal length in $W_{\lambda+\rho}wW_{\mu+\rho}$.

(e) *If μ is a dominant integral weight then*

$$\mathcal{L}^{\lambda/\mu} \cong \left\{ v \in (V^{\otimes k})_{\lambda-\mu} \mid X_i^{\langle \mu+\rho, \alpha_i^\vee\rangle} v = 0, \text{for all } 1 \le i \le n. \right\}.$$

Proof. (a) Let $v^+_{w\circ\mu}$ be the generating highest weight vector of $M(w\circ\mu)$ and, for $n \in V^{\otimes k}$ let $pr(v^+_{w\circ\mu}\otimes n)$ be the image of $v^+_{w\circ\mu}\otimes n$ in $(M\otimes V^{\otimes k})/\mathfrak{n}^-(M\otimes V^{\otimes k})$. Then, since λ is integrally dominant, Lemma 4.2 shows that

$$\begin{array}{ccc} (V^{\otimes k})_{\lambda-w\circ\mu} & \longrightarrow & \mathcal{M}^{\lambda/(w\circ\mu)} \\ n & \longmapsto & pr(v^+_{w\circ\mu}\otimes n) \end{array} \tag{4.4}$$

is a vector space isomorphism. This is a $\mathcal{B}_k$-module isomorphism since the $\mathcal{B}_k$ action on $M(w\circ\mu)\otimes V^{\otimes k}$ commutes with $\mathfrak{n}^-$ and fixes $v^+_{w\circ\mu}$.

(b) By (a), $\lambda - w\circ\mu \in P$ and so $W^\lambda = W^\mu$. It is sufficient to show that $\mathcal{M}^{\lambda/w\circ\mu} \cong \mathcal{M}^{\lambda/(s_iw\circ\mu)}$ for all simple reflections of W^λ such that $s_i \in W_{\lambda+\rho}$ and $s_iw > w$. Applying the exact functor F_λ to the Verma module inclusion

$$M(s_iw\circ\mu) \hookrightarrow M(w\circ\mu) \qquad \text{gives} \qquad \mathcal{M}^{\lambda/s_iw\circ\mu} \hookrightarrow \mathcal{M}^{\lambda/w\circ\mu},$$

an inclusion of $\tilde{\mathcal{B}}_k$-modules. Since $s_i(\lambda - w\circ\mu) = s_i(\lambda+\rho) - s_iw(\mu+\rho) = \lambda+\rho-s_iw(\mu+\rho) = \lambda - (s_iw)\circ\mu$ there is a (vector space) isomorphism of weight spaces

$$(V^{\otimes})_{\lambda-w\circ\mu} \cong V^{\otimes k}_{\lambda-s_iw\circ\mu}.$$

(This isomorphism can be realized by Lusztig's braid group action [CP, §8.1-8.2] $T_i\colon (V^{\otimes k})_{\lambda-w\circ\mu} \to (V^{\otimes k})_{s_i(\lambda-w\circ\mu)}$). Thus, by part (a), the $\tilde{\mathcal{B}}_k$-module inclusion $\mathcal{M}^{\lambda/s_iw\circ\mu} \hookrightarrow \mathcal{M}^{\lambda/w\circ\mu}$ is an isomorphism.

(c) Use the notations for the bilinear forms on $M(w\circ\mu)$ and $V^{\otimes k}$ as given in the paragraph before the statement of the proposition. Let $\{b_i\}$ be an orthonormal basis of $V^{\otimes k}$ with respect to $\langle,\rangle_{V^{\otimes k}}$. If $r \in \mathrm{rad}\langle,\rangle_M$ then

$$\langle r\otimes b, s\otimes b'\rangle = \langle r,s\rangle_M\langle b,b'\rangle_{V^{\otimes k}} = 0, \qquad \text{for all } s\in M(w\circ\mu),\ b,b'\in V^{\otimes k},$$

and so $(\mathrm{rad}\langle,\rangle_M)\otimes V^{\otimes k} \subseteq \mathrm{rad}\langle,\rangle$. Conversely, if $r_i \in M(w\circ\mu)$ such that $\sum r_i\otimes b_i \in \mathrm{rad}\langle,\rangle$ then

$$0 = \Big\langle \sum_i r_i\otimes b_i, s\otimes b_j\Big\rangle = \sum_i \langle r_i,s\rangle_M\delta_{ij} = \langle r_i, s\rangle, \qquad \text{for all } s\in M(w\circ\mu).$$

So $r_i \in \mathrm{rad}\langle,\rangle_M$ and thus $\mathrm{rad}\langle,\rangle \subseteq \mathrm{rad}\langle,\rangle_M\otimes V^{\otimes k}$. By the $U_h\mathfrak{g}$ contravariance of $\langle,\rangle$

$$(M(w\circ\mu)\otimes V^{\otimes k})^{[\lambda]}_\lambda \perp (M(w\circ\mu)\otimes V^{\otimes k})^{[\mu]}_\mu$$

for integrally dominant weights λ, μ with $\lambda\neq\mu$. Thus

$$\mathrm{rad}\langle,\rangle = \big(\mathrm{rad}\langle,\rangle_M\otimes V^{\otimes k}\big)^{[\lambda]}_\lambda, \tag{4.5}$$

where $\langle , \rangle$ is the restriction of the form on $M(w\circ\mu)\otimes V^{\otimes k}$ to $(M(w\circ\mu)\otimes V^{\otimes k})^{[\lambda]}_{\lambda}$. Thus

$$\frac{(M(w\circ\mu)\otimes V^{\otimes k})^{[\lambda]}_{\lambda}}{\mathrm{rad}\langle , \rangle} = \frac{(M(w\circ\mu)\otimes V^{\otimes k})^{[\lambda]}_{\lambda}}{(\mathrm{rad}\langle , \rangle_M \otimes V^{\otimes k})^{[\lambda]}_{\lambda}}$$
$$\cong \left(\frac{M(w\circ\mu)}{\mathrm{rad}\langle , \rangle_M}\otimes V^{\otimes k}\right)^{[\lambda]}_{\lambda} = (L(w\circ\mu)\otimes V^{\otimes k})^{[\lambda]}_{\lambda} = \mathcal{L}^{\lambda/(w\circ\mu)},$$

where the isomorphism is a consequence of the fact that, because λ is an integrally dominant weight, the functor $(\cdot\otimes V^{\otimes k})^{[\lambda]}_{\lambda}$ is exact (Prop. 3.9).

(d) If $\lambda - w\circ\mu$ is not a weight of $V^{\otimes k}$ then, by part (a), $\mathcal{M}^{\lambda/w\circ\mu}=0$. Since the functor F_λ is exact and $L(w\circ\mu)$ is a quotient of $M(w\circ\mu)$, $\mathcal{L}^{\lambda/w\circ\mu}$ is a quotient of $\mathcal{M}^{\lambda/w\circ\mu}$. Thus $\mathcal{M}^{\lambda/w\circ\mu}=0$ implies $\mathcal{L}^{\lambda/w\circ\mu}=0$.

Assume that $w\in W^\mu$ is not the longest element of the double coset $W_{\lambda+\rho}wW_{\mu+\rho}\in W^\mu$. Then there is a positive root $\alpha>0$ such that $s_\alpha\in W_{\lambda+\rho}$ and $s_\alpha w>w$. Since $s_\alpha wW_{\mu+\rho}\neq wW_{\mu+\rho}$ there is an inclusion of Verma modules $M(s_\alpha w\circ\mu)\subseteq M(w\circ\mu)$ and $F_\lambda(L(w\circ\mu))$ is a quotient of $F_\lambda(M(w\circ\mu))/F_\lambda(M(s_\alpha w\circ\mu))$. On the other hand, by part (b),

$$\mathcal{M}^{\lambda/s_\alpha w\circ\mu}\cong\mathcal{M}^{\lambda/w\circ\mu},\qquad\text{and so}\qquad \frac{\mathcal{M}^{\lambda/w\circ\mu}}{\mathcal{M}^{\lambda/s_\alpha w\circ\mu}}=\frac{F_\lambda(M(w\circ\mu))}{F_\lambda(M(s_\alpha w\circ\mu))}=0.$$

Thus $F_\lambda(L(w\circ\mu))=0$.

(e) When μ is a dominant integral weight

$$\mathrm{rad}\langle , \rangle_M = U_h\mathfrak{n}^-\{Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}v_\mu^+ \mid 1\le i\le n\} = \sum_i U_h\mathfrak{n}^- Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}v_\mu^+,$$

see [Dx, 7.2.7]. Thus, by (c) and the vector space isomorphism (4.4) it follows that, as vector spaces,

$$\mathcal{L}^{\lambda/\mu}\cong\Big(\big(\text{span-}\{pr(v_\mu^+\otimes n)\mid n\in V^{\otimes k}\}\big)\Big/\big(\text{span-}\{pr(Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}v_\mu^+\otimes n)\mid n\in V^{\otimes k}\}\big)\Big)_\lambda.$$

For any $k\ge 0$, there is a nonzero constant c such that $pr(Y_i^{k+1}v_\mu^+\otimes n)=c\cdot pr(Y_i(Y_i^k v_\mu^+\otimes n)-Y_i^k v_\mu^+\otimes Y_i n)=-c\,pr(Y_i^k v_\mu^+\otimes Y_i n)$, and so, by induction, $pr(Y_i^{k+1}v_\mu^+\otimes n)=\xi\cdot pr(v_\mu^+\otimes Y_i^{k+1}n)$ for some constant $\xi\neq 0$. Thus $\mathcal{L}^{\lambda/\mu}$ is isomorphic to the vector space

$$\left(V^{\otimes k}\Big/\Big(\sum_i Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}V^{\otimes k}\Big)\right)_{\lambda-\mu}\cong\left(\sum_i Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}V^{\otimes k}\right)^\perp_{\lambda-\mu}.$$

If $b\in(Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}V^{\otimes k})^\perp$ then the $U_h\mathfrak{g}$ contravariance of $\langle , \rangle_{V^{\otimes k}}$ gives that

$$0=\big\langle Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}n,b\big\rangle_{V^{\otimes k}}=\big\langle n,X_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}b\big\rangle_{V^{\otimes k}},\qquad\text{for all } n\in V^{\otimes k}.$$

Thus, by the nondegeneracy of $\langle , \rangle_{V^{\otimes k}}$,

$$\mathcal{L}^{\lambda/\mu}\cong\left(\sum_i Y_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}V^{\otimes k}\right)^\perp_{\lambda-\mu}=\{b\in(V^{\otimes k})_{\lambda-\mu}\mid X_i^{\langle\mu+\rho,\alpha_i^\vee\rangle}b=0\}.\qquad\blacksquare$$

Remark 4.6. Proposition 4.3d gives a necessary condition on $\lambda/w\circ\mu$ for the $\tilde{\mathcal{B}}_k$-module $\mathcal{L}^{\lambda/w\circ\mu}$ to be nonzero. The following will be useful for analyzing the combinatorics of the examples in Section 6. If P denotes the weight lattice λ is an integrally dominant then the action of $W_{\lambda+\rho}$ on $\lambda - P$ by the dot action has fundamental domain

$$C^-_{\lambda+\rho} = \{\mu \in \lambda - P \mid \langle \mu+\rho, \alpha^\vee\rangle \in \mathbb{Z}_{\leq 0} \text{ for all } \alpha > 0 \text{ such that } \langle \lambda+\rho, \alpha^\vee\rangle = 0\}.$$

and the following are equivalent:

(a) $\mu \in C^-_{\lambda+\rho}$,
(b) $\mu = w^\lambda \circ \nu$ with ν integrally dominant and $w^\lambda \in W^\lambda$ longest in the coset $W_{\lambda+\rho}w^\lambda$ in W^λ,
(c) $\mu = w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}$ with $w^\lambda_{\tilde{\mu}} \in W^\lambda$ longest in the double coset $W_{\lambda+\rho}w^\lambda_{\tilde{\mu}}W_{\mu+\rho}$ in W^λ.

In the classical case, when $\mathfrak{g}$ is type A_n and $V = L(\omega_1)$ is the $n+1$ dimensional fundamental representation the $\tilde{\mathcal{B}}_k$-module $\mathcal{L}^{\lambda/w\circ\mu}$ is a simple $\tilde{\mathcal{B}}_k$-module whenever it is nonzero (see [Su]). As the following Proposition shows, this is a very special phenomenon.

Proposition 4.7. *Assume that $V = L(\nu)$ for a dominant integral weight ν. If the $\tilde{\mathcal{B}}_k$-module $\mathcal{L}^{\lambda/\mu}$ is irreducible (or 0) for all k, all dominant integral weights μ, and all integrally dominant weights λ then*

(a) $\mathfrak{g}$ is type A_n, B_n, C_n or G_2 and $V = L(\omega_1)$, and
(b) the action of the subgroup $\mathcal{B}_k$ of $\tilde{\mathcal{B}}_k$ generates $End_{U_h\mathfrak{g}}(V^{\otimes k})$.

Proof. (a) If μ is large dominant integral weight (for example, we may take $\mu = n\rho$, $n >> 0$) then, as a $U_h\mathfrak{g}$-module,

$$L(\mu)\otimes V \cong \bigoplus_b L(\mu + \mathrm{wt}(b)),$$

where the sum is over a basis of V consisting of weight vectors and $\mathrm{wt}(b)$ is the weight of the vector b. The group $\tilde{\mathcal{B}}_1$ is generated by the element X^{ε_1} which acts on a summand $L(\lambda)$ in $L(\mu)\otimes V$ by the constant $q^{\langle\lambda,\lambda+2\rho\rangle-\langle\mu,\mu+2\rho\rangle-\langle\nu,\nu+2\rho\rangle}$. Then $F_\lambda(L(\mu))$ is the $L(\lambda)$-isotypic component of $L(\mu)\otimes V$ and these are simple $\tilde{\mathcal{B}}_1$ modules (for the various λ) only if all the values

$$\begin{aligned}\langle \mu+\mathrm{wt}(b), \mu+\mathrm{wt}(b)+2\rho\rangle - \langle\mu,\mu+2\rho\rangle - \langle\nu,\nu+2\rho\rangle \\ = 2\langle\mu+\rho,\mathrm{wt}(b)\rangle + \langle\mathrm{wt}(b),\mathrm{wt}(b)\rangle - \langle\nu,\nu+2\rho\rangle,\end{aligned} \tag{4.8}$$

as b ranges over a weight basis of V, are distinct. It follows that all weight spaces of V must be one dimensional. This means that

(a) $\mathfrak{g}$ is type A_n, B_n, C_n, D_n, E_6, E_7 or G_2 and $V = L(\omega_1)$, or
(b) $\mathfrak{g}$ is type A_n and $V = L(k\omega_1)$ or $V = (k\omega_n)$ for some k, or
(c) $\mathfrak{g}$ is type B_n and $V = L(\omega_n)$, or
(d) $\mathfrak{g}$ is type D_n and $V = L(\omega_{n-1})$ or $V = L(\omega_n)$.

Most of the weights of these representations lie in a single W-orbit. If γ and γ' are two distinct weights of V which are in the same W-orbit then $\langle\gamma,\gamma\rangle = \langle\gamma',\gamma'\rangle$. If $\mu = n\rho$ with $n >> 0$ then the condition that all the values in (4.9) be distinct forces that

$$2(n+1)\langle\rho,\gamma\rangle = 2\langle\mu+\rho,\gamma\rangle \quad\neq\quad 2\langle\mu+\rho,\gamma'\rangle = 2(n+1)\langle\rho,\gamma'\rangle.$$

Writing $\gamma = \nu - \sum_i c_i\alpha_i$ and $\gamma' = \nu - \sum_i c'_i\alpha_i$ with $c_i, c'_i \in \mathbb{Z}_{>0}$ the last equation becomes

$$2(n+1)\cdot\sum_i c_i \neq 2(n+1)\cdot\sum_i c'_i.$$

Finally, an easy case by case check verifies that the only choices of V in (a-d) above which satisfy this last condition for all weights in the W-orbit of the highest weight are those listed in the statement of the proposition.

(b) Let $\mathcal{Z}_k = \mathrm{End}_{U_h\mathfrak{g}}(V^{\otimes k})$. As a $(U_h\mathfrak{g}, \mathcal{Z}_k)$-bimodule

$$V^{\otimes k} \cong \bigoplus_{\lambda} L(\lambda) \otimes \mathcal{Z}_k^{\lambda},$$

where $\mathcal{Z}_k^{\lambda}$ is an irreducible $\mathcal{Z}_k$-module and the sum is over all dominant integral weights for which the irreducible $U_h\mathfrak{g}$-module $L(\lambda)$ appears in $V^{\otimes k}$. By restriction $\mathcal{Z}_k^{\lambda}$ is an $\mathcal{B}_k$-module and this is the $\tilde{\mathcal{B}}_k$-module $F_\lambda(L(0))$ which, by assumption, is simple. Since $L(0)$ is the trivial module X^{ε_1} acts on $F_\lambda(L(0))$ by the identity and so $F_\lambda(L(0))$ is simple as a $\mathcal{B}_k$-module. Thus the simple $\mathcal{Z}_k$-modules in $V^{\otimes k}$ coincide exactly with the simple $\mathcal{B}_k$-modules in $V^{\otimes k}$ and it follows that $\mathcal{B}_k$ generates $\mathcal{Z}_k = \mathrm{End}_{U_h\mathfrak{g}}(V^{\otimes k})$. ∎

Jantzen filtrations for affine braid group representations

Applying the functor F_λ to the Jantzen filtration of $M(\mu)$ produces a filtration of $\mathcal{M}^{\lambda/\mu}$,

$$\mathcal{M}^{\lambda/\mu} = F_\lambda(M(\mu)) = F_\lambda(M(\mu)^{(0)}) \supseteq F_\lambda(M(\mu)^{(1)}) \supseteq \cdots. \tag{4.9}$$

An argument of Suzuki [Su, Thm. 4.3.5] shows that this filtration can be obtained directly from the $\tilde{\mathcal{B}}_k$-contravariant form $\langle , \rangle_t$ on

$$\mathcal{M}^{\lambda+t\delta/\mu+t\delta} = F_{\lambda+t\delta}(M(\mu+t\delta)) = (M(\mu+t\delta)\otimes V^{\otimes k})^{[\lambda+t\delta]}_{\lambda+t\delta}$$

which is the restriction of the $U_h\mathfrak{g}$ contravariant form $\langle , \rangle_t$ on $(M(\mu+t\delta)\otimes V^{\otimes k})$, see (2.5) and (3.7). To do this define

$$\mathcal{M}^{\lambda+t\delta/\mu+t\delta}(j) = \{m \in \mathcal{M}^{(\lambda+t\delta)/(\mu+t\delta)} \mid \langle m, n\rangle_t \in t^j\mathbb{C}[t] \text{ for all } n \in \mathcal{M}^{\lambda+t\delta/\mu+t\delta}\}$$

and

$$\left(\mathcal{M}^{\lambda/\mu}\right)^{(j)} = \text{image of } \mathcal{M}^{\lambda+t\delta/\mu+t\delta}(j) \text{ in } \mathcal{M}^{\lambda+t\delta/\mu+t\delta} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/t\mathbb{C}[t]$$

to obtain a filtration

$$\mathcal{M}^{\lambda/\mu} = \left(\mathcal{M}^{\lambda/\mu}\right)^{(0)} \supseteq \left(\mathcal{M}^{\lambda/\mu}\right)^{(1)} \supseteq \cdots \tag{4.10}$$

such that the quotients $\left(\mathcal{M}^{\lambda/\mu}\right)^{(j)})/\left(\mathcal{M}^{\lambda/\mu}\right)^{(j+1)})$ carry nondegenerate $\tilde{\mathcal{B}}_k$ contravariant forms. Since, for different λ, the subspaces $(M(\mu+t\delta)\otimes V^{\otimes k})^{[\lambda+t\delta]}_{\lambda+t\delta}$ are mutually orthogonal with respect to the $U_h\mathfrak{g}$ contravariant form $\langle , \rangle_t$ on $(M(\mu+t\delta)\otimes V^{\otimes k})$,

$$(M(\mu+t\delta)(j)\otimes V^{\otimes k})^{[\lambda+t\delta]}_{\lambda+t\delta} \subseteq (M(\mu+t\delta)\otimes V^{\otimes k})^{[\lambda+t\delta]}_{\lambda+t\delta}(j) = \mathcal{M}^{(\lambda+t\delta)/(\mu+t\delta)}(j).$$

On the other hand, if $u \in (M(\mu+t\delta)\otimes V^{\otimes k})^{[\lambda+t\delta]}_{\lambda+t\delta}(j)$ then write $u = \sum_i a_i\otimes b_i$ where $a_i \in M(\mu+t\delta)$ and b_i is an orthonormal basis of $V^{\otimes k}$. Then, for all $v \in M(\mu+t\delta)$, and all k, $\langle a_k, v\rangle_t = \langle u, v\otimes b_k\rangle_t \in t^j\mathbb{C}[t]$ and so $u \in (M(\mu+t\delta)(j)\otimes V^{\otimes k})^{[\lambda+t\delta]}_{\lambda+t\delta}$. So

$$F_{\lambda+t\delta}(M(\mu+t\delta))^{(j)} = \mathcal{M}^{(\lambda+t\delta)/(\mu+t\delta)}(j)$$

and the filtrations in (4.10) and (4.11) are identical.

Proposition 4.11. *Let λ and μ be integrally dominant weights and let $w, y \in W^\mu$ be elements of maximal length in $W_{\lambda+\rho} w W_{\mu+\rho}$ and $W_{\lambda+\rho} y W_{\mu+\rho}$ respectively. Assume that the $\tilde{\mathcal{B}}_k$ modules $\mathcal{L}^{\lambda/v\circ\mu}$, $v \in W^\mu$, are simple. Then multiplicities of $\mathcal{L}^{\lambda/y\circ\mu}$ in the filtration (4.11) are given by*

$$\sum_{j\geq 0} \left[\frac{(\mathcal{M}^{\lambda/w\circ\mu})^{(j)}}{(\mathcal{M}^{\lambda/w\circ\mu})^{(j+1)}} : \mathcal{L}^{\lambda/(y\circ\mu)} \right] \mathtt{v}^{\frac{1}{2}(\ell(y)-\ell(w)-j)} = P_{wy}(\mathtt{v}).$$

where $P_{wy}(\mathtt{v})$ is the Kazhdan-Lusztig polynomial for the Weyl group W^μ.

Proof. Since the functor F_λ is exact this result follows from the Beilinson-Bernstein theorem (2.6). The condition on y is necessary for the module $\mathcal{L}^{\lambda/y\circ\mu}$ to be nonzero. ∎

The BGG resolution for affine braid groups

Let $\mu \in \mathfrak{h}^*$ be such that $-(\mu+\rho)$ is dominant and regular and let W_J^μ be a parabolic subgroup of the integral Weyl group W^μ. Let w_0 be the longest element of W_J^μ and fix $\nu = w_0 \circ \mu$. Following the method of [Ch2], applying the exact functor F_λ to the BGG resolution in (2.7) produces an exact sequence of $\tilde{\mathcal{B}}_k$-modules

$$0 \to \mathcal{C}_N \longrightarrow \cdots \longrightarrow \mathcal{C}_1 \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{L}^{\lambda/\nu} \longrightarrow 0 \qquad \text{where} \qquad \mathcal{C}_k = \bigoplus_{\ell(w)=j} \mathcal{M}^{\lambda/w\circ\nu}, \tag{4.12}$$

and the sum is over all $w \in W_J^\mu$ of length j (in W_J^μ). Thus, in the Grothendieck group of the category $\tilde{\mathcal{O}}_k$ of finite dimensional $\tilde{\mathcal{B}}_k$-modules

$$[\mathcal{L}^{\lambda/\nu}] = \sum_{w\in W_J^\mu} (-1)^{\ell(w)} [\mathcal{M}^{\lambda/(w\circ\nu)}] \tag{4.13}$$

where $\nu = w_0 \circ \mu$ and w_0 is the longest element of W_J^μ. This identity is a generalization of the classical Jacobi-Trudi identity [Mac I (5.4)] for expanding Schur functions in terms of homogeneous symmetric functions

$$s_{\lambda/\nu} = \sum_{w\in S_n} (-1)^{\ell(w)} h_{\lambda+\delta-w(\nu+\delta)}. \tag{4.14}$$

Restriction of $\mathcal{L}^{\lambda/\mu}$ to the braid group

The braid group $\mathcal{B}_k$ is the quotient of the affine braid group by the relation $X^{\varepsilon_1} = 1$ and so the modules $\mathcal{L}^{\nu/0}$ are $\mathcal{B}_k$-modules. The following proposition determines the structure of $F_\lambda(L(\mu))$ as a $\mathcal{B}_k$ module when $L(\mu)$ is finite dimensional. This is a generalization of the Littlewood-Richardson rule.

Proposition 4.15. *Let P^+ be the set of dominant integral weights. Define the tensor product multiplicities $c_{\mu\nu}^\lambda$, $\lambda, \mu, \nu \in P^+$, by the $U_h\mathfrak{g}$-module decompositions*

$$L(\mu) \otimes L(\nu) \cong \bigoplus_{\lambda\in P^+} L(\lambda)^{\oplus c_{\mu\nu}^\lambda}.$$

Then

$$Res^{\tilde{\mathcal{B}}_k}_{\mathcal{B}_k}(\mathcal{L}^{\lambda/\mu}) = \bigoplus_{\nu\in P^+} (\mathcal{L}^{\nu/0})^{\oplus c^\lambda_{\mu\nu}}.$$

Proof. Let us abuse notation slightly and write sums instead of direct sums. Then, as a $(U_h\mathfrak{g}, \mathcal{B}_k)$ bimodule

$$L(\mu)\otimes V^{\otimes k} = \sum_\lambda L(\lambda)\otimes \mathcal{L}^{\lambda/\mu},$$

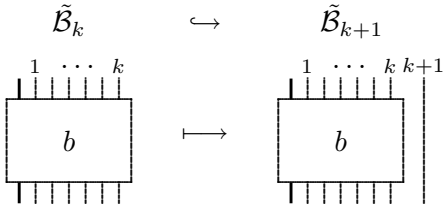
where $\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu))$. As a $(U_h\mathfrak{g}, \mathcal{B}_k)$ bimodule

$$L(\mu)\otimes V^{\otimes k} = L(\mu)\otimes\left(\sum_\nu L(\nu)\otimes\mathcal{L}^{\nu/0}\right) = \sum_{\lambda,\nu} c^\lambda_{\mu\nu} L(\lambda)\otimes\mathcal{L}^{\nu/0}.$$

Comparing coefficients of $L(\lambda)$ in these two identities yields the formula in the statement. ∎

5. Markov traces

A Markov trace on the affine braid group is a trace functional which respects the inclusions $\tilde{\mathcal{B}}_1 \subseteq \tilde{\mathcal{B}}_2 \subseteq \cdots$ where



(5.1)

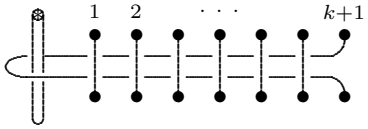
More precisely, a *Markov trace* on the affine braid group with parameters $z, Q_1, Q_2, \ldots \in \mathbb{C}$ is a sequence of functions

$$\mathrm{mt}_k\colon \tilde{\mathcal{B}}_k \longrightarrow \mathbb{C} \qquad \text{such that}$$

(1) $\mathrm{mt}_1(1) = 1$,
(2) $\mathrm{mt}_{k+1}(b) = \mathrm{mt}_k(b)$, for $b \in \tilde{\mathcal{B}}_k$,
(3) $\mathrm{mt}_k(b_1b_2) = \mathrm{mt}_k(b_2b_1)$, for $b_1, b_2 \in \tilde{\mathcal{B}}_k$,
(4) $\mathrm{mt}_{k+1}(bT_k) = z\mathrm{mt}_k(b)$, for $b \in \tilde{\mathcal{B}}_k$,
(5) $\mathrm{mt}_{k+1}(b(\tilde{X}^{\varepsilon_{k+1}})^r) = Q_r\mathrm{mt}_k(b)$, for $b \in \tilde{\mathcal{B}}_k$,

where

$$\tilde{X}^{\varepsilon_{k+1}} = T_kT_{k-1}\cdots T_2X^{\varepsilon_1}T_2^{-1}\cdots T_{k-1}^{-1}T_k^{-1} =$$



If M is a finite dimensional $U = U_h\mathfrak{g}$ module and $a \in \mathrm{End}_U(M)$ the *quantum trace* of a on M (see [LR §3] and [CP Def. 4.2.9]) is the trace of the action of $e^{h\rho}a$ on M,

$$\mathrm{tr}_q(a) = \mathrm{Tr}(e^{h\rho}a, M), \qquad \text{and} \qquad \dim_q(M) = \mathrm{tr}_q(\mathrm{id}_M) = \mathrm{Tr}(e^{h\rho}, M) \tag{5.2}$$

is the *quantum dimension* of M. The first step of the standard argument for proving Weyl's dimension formula [B-tD, VI Lemma 1.19] shows that the quantum dimension of the finite dimensional irreducible $U_h\mathfrak{g}$-module $L(\mu)$ is

$$\dim_q(L(\mu)) = \mathrm{Tr}(e^{h\rho}, L(\mu)) = \prod_{\alpha>0} \frac{e^{\frac{h}{2}\langle \mu+\rho,\alpha^\vee\rangle} - e^{-\frac{h}{2}\langle \mu+\rho,\alpha^\vee\rangle}}{e^{\frac{h}{2}\langle \rho,\alpha^\vee\rangle} - e^{-\frac{h}{2}\langle \rho,\alpha^\vee\rangle}} = \prod_{\alpha>0} \frac{[\langle \mu+\rho,\alpha^\vee\rangle]}{[\langle \rho,\alpha^\vee\rangle]}, \tag{5.3}$$

where $q = e^{h/2}$ and $[d] = (q^d - q^{-d})/(q - q^{-1})$ for a positive integer d.

Theorem 5.4. *Let $\mu, \nu \in P^+$ be dominant integral weights. Let $M = L(\mu)$ and $V = L(\nu)$ and let Φ_k be the representation of $\tilde{\mathcal{B}}_k$ defined in Proposition 3.5. Then the functions*

$$\begin{array}{rccc} mt_k\colon & \tilde{\mathcal{B}}_k & \longrightarrow & \mathbb{C} \\ & b & \longmapsto & \dfrac{tr_q(\Phi_k(b))}{\dim_q(M)\dim_q(V)^k} \end{array}$$

form a Markov trace on the affine braid group with parameters

$$z = \frac{q^{\langle \nu,\nu+2\rho\rangle}}{\dim_q(V)} \quad \text{and} \quad Q_r = \sum_\lambda q^{r(\langle \lambda,\lambda+2\rho\rangle - \langle \mu,\mu+2\rho\rangle - \langle \nu,\nu+2\rho\rangle)} \frac{\dim_q(L(\lambda))c^\lambda_{\mu\nu}}{\dim_q(L(\mu))\dim_q(L(\nu))},$$

where the positive integers $c^\lambda_{\mu\nu}$ and the sum in the expression for Q_r are as in the tensor product decomposition

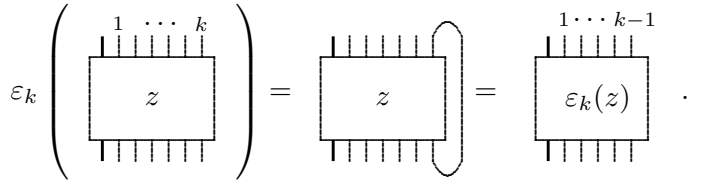
$$L(\mu)\otimes L(\nu) = \bigoplus_\lambda L(\lambda)^{\oplus c^\lambda_{\mu\nu}}.$$

Proof. The fact that mt_k as defined in the statement of the Theorem satisfies (1)-(4) in the definition of a Markov trace follows exactly as in [LR] Theorem 3.10c. The formula for the parameter z is derived in [LR, (3.9) and Thm. 3.10(2)].

It remains to check (5). The proof is a combination of the argument used in [Or] Theorem 5.3 and the argument in the proof of [LR] Theorem 3.10c. Let $\varepsilon_k\colon \mathrm{End}_U(M\otimes V^{\otimes k}) \to \mathrm{End}_U(M\otimes V^{\otimes(k-1)})$ be given by

$$\varepsilon_k(z) = (\mathrm{id}_{M\otimes V^{\otimes(k-1)}}\otimes \check{e})\circ(z\otimes \mathrm{id}) \qquad \text{where} \qquad \begin{array}{rccc} \check{e}\colon & V\otimes V^* & \longrightarrow & \mathbb{C} \\ & x\otimes\phi & \longmapsto & \dim_q(V)^{-1}\phi(e^{h\rho}x). \end{array} \tag{5.5}$$

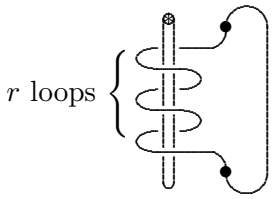
If V is simple then $\check{e}$ is the unique $U_h\mathfrak{g}$-invariant projection onto the invariants in $V\otimes V^*$. Pictorially,



The argument of [LR] Theorem 3.10b shows that

$$\mathrm{mt}_k(b) = \mathrm{mt}_{k-1}(\varepsilon_{k-1}(b)), \qquad \text{if } b\in\tilde{\mathcal{B}}_k. \tag{5.6}$$

Since $\varepsilon_1((X^{\varepsilon_1})^r)$ is a $U_h\mathfrak{g}$-module homomorphism from M to M and, since M is simple, Schur's lemma implies that

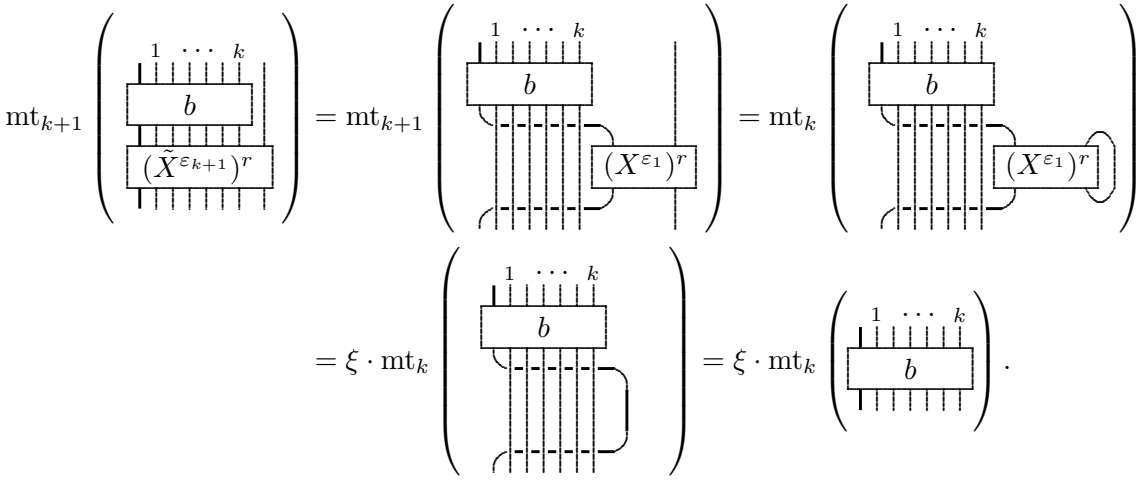


$$= \varepsilon_1((X^{\varepsilon_1})^r) = \xi \cdot \mathrm{id}_M, \qquad \text{for some } \xi \in \mathbb{C}.$$

Let $\tilde{R}_i = \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VM} \otimes \mathrm{id}_V^{(k+1)-i}$. Then $(\tilde{X}^{\varepsilon_{k+1}})^r = (\tilde{R}_k \cdots \tilde{R}_1)^{-1}(X^{\varepsilon_1})^r(\tilde{R}_k \cdots \tilde{R}_1)$ and

$$\begin{aligned}
\mathrm{mt}_{k+1}(b(\tilde{X}^{\varepsilon_{k+1}})^r) &= \mathrm{mt}_k(\varepsilon_k(b(\tilde{X}^{\varepsilon_{k+1}})^r)) \\
&= \mathrm{mt}_k(\varepsilon_k(b(\tilde{R}_k \cdots \tilde{R}_1)^{-1}(X^{\varepsilon_1})^r(\tilde{R}_k \cdots \tilde{R}_1)) \\
&= \mathrm{mt}_k(b(\tilde{R}_k \cdots \tilde{R}_1)^{-1}\varepsilon_1((X^{\varepsilon_1})^r)(\tilde{R}_k \cdots \tilde{R}_1)) \\
&= \mathrm{mt}_k(b(\tilde{R}_k \cdots \tilde{R}_1)^{-1}\xi \cdot \mathrm{id}_M \tilde{R}_k \cdots \tilde{R}_1) = \xi \cdot \mathrm{mt}_k(b).
\end{aligned}$$

This last calculation is more palatable in a pictorial format,



It remains to calculate the constant ξ. By (2.14),

$$(X^{\varepsilon_1})^r = (\check{R}_0^2)^r = \left(\sum_\lambda q^{c(\lambda)} P^\lambda_{\mu\nu}\right)^r = \sum_\lambda q^{rc(\lambda)} P^\lambda_{\mu\nu},$$

where $c(\lambda) = \langle \lambda, \lambda + 2\rho\rangle - \langle \mu, \mu + 2\rho\rangle - \langle \nu, \nu + 2\rho\rangle$ and $P^\lambda_{\mu\nu}$ is the projection onto the $L(\lambda)^{\oplus c^\lambda_{\mu\nu}}$ component in the decomposition of $M \otimes V = L(\mu) \otimes L(\nu)$. Thus

$$\begin{aligned}
\xi = \mathrm{mt}_0(\xi \cdot \mathrm{id}_M) = \mathrm{mt}_1((X^{\varepsilon_1})^r) &= \frac{1}{\dim_q(M)\dim_q(V)} \mathrm{tr}_q\left(\sum_\lambda q^{rc(\lambda)} P^\lambda_{\mu\nu}\right) \\
&= \frac{1}{\dim_q(M)\dim_q(V)} \mathrm{tr}_q\left(\sum_\lambda q^{rc(\lambda)} c^\lambda_{\mu\nu} \mathrm{id}_{L(\lambda)}\right) \\
&= \sum_\lambda q^{rc(\lambda)} c^\lambda_{\mu\nu} \frac{\dim_q(L(\lambda))}{\dim_q(L(\mu))\dim_q(L(\nu))}. \qquad \blacksquare
\end{aligned}$$

Remark 5.7. There is another formula [TW, Lemma (3.51)] for the constant Q_1 in Theorem 5.4, namely,

$$Q_1 = \frac{\sum_{w\in W}(-1)^{\ell(w)}q^{\langle \mu+\rho, w(\nu+\rho)\rangle}}{\sum_{w\in W}(-1)^{\ell(w)}q^{\langle \mu+\rho, w\rho\rangle}}, \tag{5.8}$$

where W is the Weyl group of $\mathfrak{g}$.

Let mt_k be as in Theorem 5.4 and let $\tilde{\mathcal{Z}}_k = \mathrm{End}_U(M\otimes V^{\otimes k})$. Then mt_k is the restriction of the linear functional

$$\begin{array}{rccc} \mathrm{mt}_k\colon & \tilde{\mathcal{Z}}_k & \longrightarrow & \mathbb{C} \\ & a & \longmapsto & \dfrac{\mathrm{tr}_q(a)}{\dim_q(M)\dim_q(V^{\otimes k})} \end{array} \tag{5.9}$$

to $\Phi_k(\tilde{\mathcal{B}}_k)$. Since $M\otimes V^{\otimes k}$ is a finite dimensional semisimple module $\tilde{\mathcal{Z}}_k$ is a finite dimensional semisimple algebra. The *weights* of the Markov trace mt are the constants $t_{\lambda/\mu}$ defined by

$$\mathrm{mt}_k = \sum_{\lambda} t_{\lambda/\mu}\chi^{\lambda/\mu}_{\tilde{\mathcal{Z}}_k}, \tag{5.10}$$

where $\chi^{\lambda/\mu}_{\tilde{\mathcal{Z}}_k}$ are the irreducible characters of $\tilde{\mathcal{Z}}_k$.

Theorem 5.11. *Let $M = L(\mu)$ and $V = L(\nu)$ be finite dimensional irreducible $U_h\mathfrak{g}$-modules. The weights of the Markov trace on the affine braid group defined in Theorem 5.4 are*

$$t_{\lambda/\mu} = \frac{\dim_q(L(\lambda))}{\dim_q(L(\mu))\dim_q(V)^k}.$$

Proof. Since $M\otimes V^{\otimes k}$ is finite dimensional and semisimple the algebra $\tilde{\mathcal{Z}}_k = \mathrm{End}_U(M\otimes V^{\otimes k})$ is a finite dimensional semisimple algebra. Schur's lemma can be used to show that, as a $(U_h\mathfrak{g}, \tilde{\mathcal{Z}}_k)$ bimodule,

$$M\otimes V^{\otimes k} \cong \bigoplus_{\lambda} L(\lambda)\otimes \mathcal{L}^{\lambda/\mu}, \tag{5.12}$$

where the $\mathcal{L}^{\lambda/\mu}$ are the irreducible $\tilde{\mathcal{Z}}_k$ modules. In the notation of (4.1), $\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu))$ and $\chi^{\lambda/\mu}_{\tilde{\mathcal{Z}}_k}$ is the character of $\mathcal{L}^{\lambda/\mu}$. Taking the quantum trace on both sides of (5.12) gives

$$\mathrm{tr}_q(a) = \mathrm{Tr}(e^{-h\rho}a) = \sum_{\lambda}\mathrm{Tr}(e^{-h\rho}, L(\lambda))\chi^{\lambda/\mu}_{\tilde{\mathcal{Z}}_k}(a) = \sum_{\lambda}\mathrm{tr}_q(L(\lambda))\chi^{\lambda/\mu}_{\tilde{\mathcal{Z}}_k}(a).$$

The result follows by dividing both sides by $\dim_q(L(\mu))\dim_q(V)^k$. ∎

6. Examples

Affine and cyclotomic Hecke algebras

Let $q \in \mathbb{C}^*$ be transcendental (so that we may view it as a variable when necessary). The *affine Hecke algebra* $\tilde{H}_k$ is the quotient of the group algebra $\mathbb{C}\tilde{\mathcal{B}}_k$ of the affine braid group by the relations

$$T_i^2 = (q - q^{-1})T_i + 1, \qquad 1 \le i \le k-1. \tag{6.1}$$

The affine Hecke algebra $\tilde{H}_k$ is an infinite dimensional algebra with a very interesting representation theory (see [KL] and [CG]). With X as in (3.4) the subalgebra

$$\mathbb{C}[X] = \mathbb{C}[X^{\pm\varepsilon_1}, \dots, X^{\pm\varepsilon_k}] = \text{span } \{X^\lambda \mid \lambda \in L\}$$

is a commutative subalgebra of $\tilde{H}_k$. It is a theorem of Bernstein and Zelevinsky (see [RR, Theorem 4.12]) that the center of $\tilde{H}_k$ is the ring of symmetric (Laurent) polynomials in $X^{\pm\varepsilon_1}, \dots, X^{\pm\varepsilon_k}$,

$$Z(\tilde{H}_k) = \mathbb{C}[X]^{S_k} = \mathbb{C}[X^{\pm\varepsilon_1}, \dots, X^{\pm\varepsilon_k}]^{S_k}.$$

If $w \in S_k$ define $T_w = T_{i_1} \cdots T_{i_p}$ if $w = s_{i_1} \cdots s_{i_p}$ is a reduced word for w in terms of the generating reflections $s_i = (i, i+1)$, $1 \le i \le k-1$, of S_k. Then, with X^λ as in (3.4)

$$\{X^\lambda T_w \mid \lambda \in L, w \in S_k\} \quad \text{is a basis of } \tilde{H}_k.$$

Let $u_1, \dots, u_r \in \mathbb{C}$. The *cyclotomic Hecke algebra* $H_{r,1,n}$ with parameters $u_1, \dots, u_r, q$ is the quotient of the affine Hecke algebra by the relation

$$(X^{\varepsilon_1} - u_1)(X^{\varepsilon_1} - u_2) \cdots (X^{\varepsilon_1} - u_r) = 0. \tag{6.2}$$

The algebra $H_{r,1,n}$ is a deformation of the group algebra of the complex reflection group $G(r,1,n) = (\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and is of dimension $r^n n!$. It was introduced by Ariki and Koike [AK] and its representations and its connection to the affine Hecke algebra have been well studied ([Ar],[AK],[Gk]).

The affine and cyclotomic BMW algebras

Fix $q, z \in \mathbb{C}^*$ and an infinite number of values $Q_1, Q_2, \dots$ in $\mathbb{C}$. The *affine BMW (Birman-Murakami-Wenzl) algebra* $\tilde{\mathcal{Z}}_k$ is the quotient of the group algebra $\mathbb{C}\tilde{\mathcal{B}}_k$ of the affine braid group by the relations

(6.3a) $(T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0,$

(6.3b) $E_i T_i^{\pm 1} = T_i^{\pm 1} E_i = z^{\mp 1} E_i,$

(6.3c) $E_i T_{i-1}^{\pm 1} E_i = z^{\pm 1} E_i \quad \text{and} \quad E_i T_{i+1}^{\pm 1} E_i = z^{\pm 1} E_i,$

(6.3d) $E_1 (X^{\varepsilon_1})^r E_1 = Q_r E_1,$

(6.3e) $E_1 X^{\varepsilon_1} T_1 X^{\varepsilon_1} = z^{-1} E_1,$

where the E_i, $1 \le i \le k-1$, are defined by the equations

$$\frac{T_i - T_i^{-1}}{q - q^{-1}} = 1 - E_i, \qquad 1 \le i \le k-1. \tag{6.4}$$

It follows that

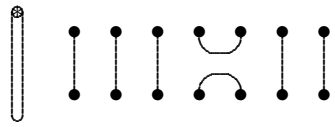
$$E_i^2 = x E_i \qquad \text{where} \qquad x = \frac{z - z^{-1}}{q - q^{-1}} + 1. \tag{6.5}$$

The classical *BMW algebra* is the subalgebra $\mathcal{Z}_k$ of the affine BMW algebra which is generated by $T_1, \dots, T_{k-1}$, and $E_1, \dots, E_{k-1}$. Fix $u_1, \dots, u_r \in \mathbb{C}$. The *cyclotomic BMW* algebra $\mathcal{Z}_{r,1,k}$ is the quotient of the affine BMW algebra by the relation

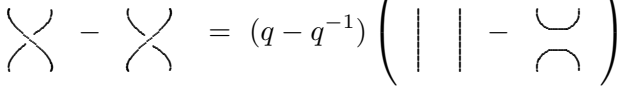
$$(X^{\varepsilon_1} - u_1)(X^{\varepsilon_1} - u_2) \cdots (X^{\varepsilon_1} - u_r) = 0. \tag{6.6}$$

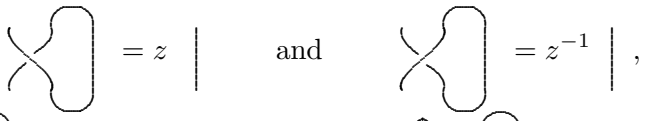
The cyclotomic BMW algebras have been defined and studied by [Hä1-3] and, for all practical purposes the affine BMW algebras appear in these papers. The cyclotomic BMW algebras are quotients of the affine BMW algebras in the same way that cyclotomic Hecke algebras are quotients of affine Hecke algebras. The classical BMW algebras $\mathcal{Z}_k = \mathcal{Z}_{1,1,k}$ have been studied in [Wz2], [HR], [Mu], [LR] and many other works. The "degenerate" version of the affine BMW algebras was defined by Nazarov [Nz] who called them "degenerate affine Wenzl algebras". The relation between his algebras and the affine BMW algebras $\tilde{\mathcal{Z}}_k$ is analogous to the relation between the graded Hecke algebras (sometimes called the degenerate affine Hecke algebras) and the affine Hecke algebras (see [Lu]).

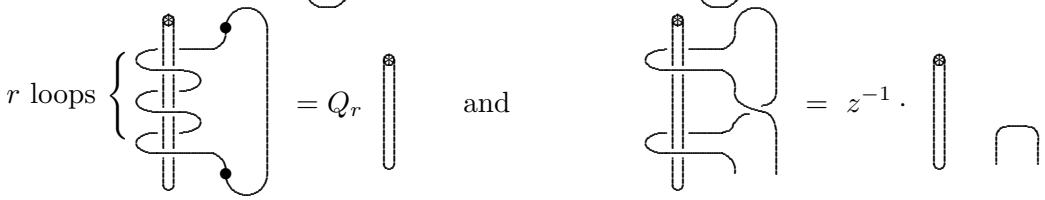
Goodman and Hauschild [GH] have proved that elements of the affine BMW algebra can be viewed as linear combinations of affine tangles. An affine tangle has k strands and a flagpole just as in the case of an affine braid, but there is no restriction that a strand must connect an upper vertex to a lower vertex. Let X^{ε_1} and T_i be the affine braids given in (3.1) and let

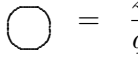
$E_i =$  (6.7)

Then $\tilde{\mathcal{Z}}_k$ is the algebra of linear combinations of tangles generated by $X^{\varepsilon_1}, T_1, \dots, T_{k-1}, E_1, \dots, E_{k-1}$ and the relations in (6.3) expressed in the form

 (6.8)

 (6.9)

 (6.10)

 (6.11)

When working with this algebra it is useful to note that

$$\begin{aligned}
T_{i-1}X^{\varepsilon_{i-1}}T_{i-1}X^{\varepsilon_{i-1}} &= X^{\varepsilon_i}X^{\varepsilon_{i-1}} = X^{\varepsilon_{i-1}}T_{i-1}X^{\varepsilon_{i-1}}T_{i-1}, \qquad \text{and, by induction,} \\
E_{i-1}X^{\varepsilon_{i-1}}T_{i-1}X^{\varepsilon_{i-1}} &= E_{i-1}T_{i-2}T_{i-1}T_{i-1}^{-1}X^{\varepsilon_{i-2}}T_{i-2}T_{i-1}X^{\varepsilon_{i-1}} \\
&= E_{i-1}E_{i-2}X^{\varepsilon_{i-2}}T_{i-2}T_{i-1}T_{i-2}^{-1}X^{\varepsilon_{i-1}} \\
&= E_{i-1}E_{i-2}X^{\varepsilon_{i-2}}T_{i-2}X^{\varepsilon_{i-2}}T_{i-1}T_{i-2} \\
&= z^{-1}E_{i-1}E_{i-2}T_{i-1}T_{i-2} = z^{-1}E_{i-1}E_{i-2}E_{i-1} = z^{-1}E_{i-1}.
\end{aligned} \tag{6.12}$$

Schur-Weyl duality for affine and cyclotomic Hecke and BMW algebras

In order to explicitly compute the representations of affine and cyclotomic Hecke algebras and BMW algebras which are obtained by applying the functors F_λ we need to fix notations for working with the representations of finite dimensional complex semisimple Lie algebras of classical type.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of type A_n, B_n, C_n or D_n and let $U_h\mathfrak{g}$ be the corresponding Drinfeld-Jimbo quantum group. Use the notations in [Bou, p. 252-258] for the root systems of types A_n, B_n, C_n and D_n so that $\varepsilon_1, \dots, \varepsilon_n$ are orthonormal (in type A_n also include ε_{n+1}),

$$\begin{aligned} \mathfrak{h}^* &= \left\{ \lambda_1\varepsilon_1 + \cdots \lambda_{n+1}\varepsilon_{n+1} \;\middle|\; \lambda_i \in \mathbb{R}, \textstyle\sum_i \lambda_i = 0 \right\}, \quad \text{in type } A_n, \quad \text{and} \\ \mathfrak{h}^* &= \left\{ \lambda_1\varepsilon_1 + \cdots \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{R} \right\}, \qquad\qquad \text{in types } B_n, C_n \text{ and } D_n, \end{aligned}$$

the fundamental weights are given by

$$\begin{aligned} \omega_i &= \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}), && 1 \le i \le n, && \text{in Type } A_n, \\ \omega_i &= \varepsilon_1 + \cdots + \varepsilon_i, && 1 \le i \le n-1, && \text{in Type } B_n, \\ \omega_n &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n), \\ \omega_i &= \varepsilon_1 + \cdots + \varepsilon_i, && 1 \le i \le n, && \text{in Type } C_n, \\ \omega_i &= \varepsilon_1 + \cdots + \varepsilon_i, && 1 \le i \le n-2, \\ \omega_{n-1} &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n), && && \text{in Type } D_n, \\ \omega_n &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n), \end{aligned}$$

and the finite dimensional $U_h\mathfrak{g}$ modules $L(\lambda)$ are indexed by dominant integral weights

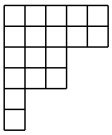
$$\begin{aligned} &\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \\ &\qquad -\tfrac{|\lambda|}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}), && \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0, \\ &&& \lambda_1, \dots, \lambda_n \in \mathbb{Z}, && \text{in Type } A_n, \\ &\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, && \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0, \\ &&& \lambda_1, \dots, \lambda_n \in \mathbb{Z}, \text{or} && \text{in Type } B_n, \\ &&& \lambda_1, \dots, \lambda_n \in \tfrac{1}{2} + \mathbb{Z}, \\ &\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, && \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0, && \text{in Type } C_n, \\ &&& \lambda_1, \dots, \lambda_n \in \mathbb{Z}, \\ &\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, && \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n-1} \ge |\lambda_n| \ge 0, \\ &&& \lambda_1, \dots, \lambda_n \in \mathbb{Z}, \text{or} && \text{in Type } D_n, \\ &&& \lambda_1, \dots, \lambda_n \in \tfrac{1}{2} + \mathbb{Z}, \end{aligned}$$

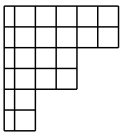
where $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

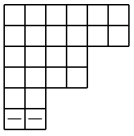
$$2\rho = \sum_{i=1}^{n} (y - 2i + 1)\varepsilon_i, \qquad \text{where} \quad y = \begin{cases} n+1, & \text{in type } A_n, \\ 2n, & \text{in type } B_n, \\ 2n+1, & \text{in type } C_n, \\ 2n-1, & \text{in type } D_n, \end{cases} \tag{6.13}$$

and, in type A_n the sum is over $1 \le i \le n+1$ instead of $1 \le i \le n$.

Identify λ with the configuration of boxes which has λ_i boxes in row i ($1 \le i \le n$). If $\lambda_i \le 0$ put $|\lambda_i|$ boxes in row i and mark them with $-$ signs. For example

$\lambda =$  $= \begin{cases} 5\varepsilon_1+5\varepsilon_2+3\varepsilon_3+3\varepsilon_4+\varepsilon_5+\varepsilon_6-\frac{18}{n+1}(\varepsilon_1+\cdots+\varepsilon_{n+1}), & \text{in type } A_n, \\ 5\varepsilon_1+5\varepsilon_2+3\varepsilon_3+3\varepsilon_4+\varepsilon_5+\varepsilon_6, & \text{in types } B_n, C_n, \text{ and } D_n, \end{cases}$

$\lambda =$  $= \frac{11}{2}\varepsilon_1+\frac{11}{2}\varepsilon_2+\frac{7}{2}\varepsilon_3+\frac{7}{2}\varepsilon_4+\frac{3}{2}\varepsilon_5+\frac{3}{2}\varepsilon_6,$ in Types B_n and D_n, and

$\lambda =$  $= 6\varepsilon_1+6\varepsilon_2+4\varepsilon_3+4\varepsilon_4+2\varepsilon_5-2\varepsilon_6,$ in Type D_6,

If b is a box in position (i,j) of λ the *content* of b is

$$c(b) = j - i = \text{the diagonal number of } b. \tag{6.14}$$

If $\lambda = \lambda_1\varepsilon_1 + \cdots \lambda_n\varepsilon_n$, then

$$\langle \lambda, \lambda+2\rho\rangle - \langle \lambda-\varepsilon_i, \lambda-\varepsilon_i+2\rho\rangle = 2\lambda_i + 2\rho_i - 1 = y + 2\lambda_i - 2i = y + 2c(\lambda/\lambda^-),$$

where λ/λ^- is the box at the end of row i in λ. Note that $c(\lambda/\lambda^-)$ may be a $\frac{1}{2}$-integer. Also, in types B_n and D_n,

$$\langle \omega_n, \omega_n+2\rho\rangle = \frac{n}{4} + \frac{1}{2}\sum_{i=1}^{n}(y-2i+1) = \frac{n}{4} + \frac{n}{2}\cdot y - \frac{n^2}{2} = \begin{cases} \frac{n^2}{2}+\frac{n}{4}, & \text{in type } B_n, \\ \frac{n^2}{2}-\frac{n}{4}, & \text{in type } D_n. \end{cases}$$

Using these formulas $\langle \lambda, \lambda+2\rho\rangle$ can easily be computed for all dominant integral weights λ. For example

$$\langle \lambda, \lambda+2\rho\rangle = y|\lambda| + 2\sum_{b\in\lambda} c(b) + \begin{cases} -\dfrac{|\lambda|^2}{n+1}, & \text{in type } A_n, \\ 0, & \text{in type } C_n \text{ or in type } B_n \text{ with } \lambda_i \in \mathbb{Z}, \\ \dfrac{n}{4}+\dfrac{n^2}{2}, & \text{in type } B_n \text{ with } \lambda_i \in \frac{1}{2}+\mathbb{Z}. \end{cases} \tag{6.15}$$

For all dominant integral weights λ in type A_n, B_n and C_n we have

$$L(\lambda)\otimes L(\omega_1) = \begin{cases} \displaystyle\bigoplus_{\lambda^+} L(\lambda^+), & \text{in type } A_n, \\ L(\lambda)\displaystyle\bigoplus\left(\bigoplus_{\lambda^\pm} L(\lambda^\pm)\right), & \text{in Type } B_n \text{ with } \lambda_n > 0, \\ \left(\displaystyle\bigoplus_{\lambda^\pm} L(\lambda^\pm)\right), & \text{in types } C_n \text{ and } D_n, \text{ and} \\ & \text{in type } B_n \text{ with } \lambda_n = 0, \end{cases} \tag{6.16}$$

where the sum over λ^+ is a sum over all partitions (of length $\leq n$) obtained by adding a box to λ, and the sum over $\lambda^\pm$ denotes a sum over all dominant weights obtained by adding or removing a box from λ. In type D_n addition and removal of a box should include the possibility of addition and removal of a box marked with a $-$ sign, and removal of a box from row n when $\lambda_n = \frac{1}{2}$ changes λ_n to $-\frac{1}{2}$.

Theorem 6.17. *Let $\mathfrak{g}$ be the simple complex Lie algebra of classical type, $U = U_h\mathfrak{g}$ the corresponding quantum group and let $V = L(\omega_1)$ be the irreducible $U_h\mathfrak{g}$ module of highest weight ω_1. For each $M \in \mathcal{O}$ let $\Phi_k \colon \tilde{\mathcal{B}}_k \to End_U(M \otimes V^{\otimes k})$ be the affine braid group representation defined in Proposition 3.5.*

(a) If $\mathfrak{g}$ is type A_n then Φ_k is a representation of the affine Hecke algebra $\tilde{H}_k$ with $q = e^{h/2}$. (In this Type A_n case use a different normalization of the map Φ_k and set $\Phi_k(T_i) = q^{1/(n+1)}\check{R}_i$.)

(b) If $\mathfrak{g}$ is type A_n and if $M = L(\mu)$ where μ is a dominant integral weight then Φ_k is a representation of the cyclotomic Hecke algebra $H_{r,1,n}(u_1, \dots, u_r)$ for any (multi)set of parameters $u_1, \dots, u_r$ containing the (multi)set of values $q^{2c(b)}$ as b runs over the addable boxes of μ.

(c) If $\mathfrak{g}$ is type B_n, C_n or D_n and M is a highest weight module then there are unique values $Q_1, Q_2, \ldots \in \mathbb{C}$, depending only on the central character of M, such that Φ_k is a representation of the affine BMW algebra $\tilde{\mathcal{Z}}_k$ with parameters $Q_1, Q_2, \ldots$,

$$q = e^{h/2}, \qquad \text{and} \qquad z = \begin{cases} q^{2n}, & \text{in Type } B_n, \\ -q^{2n+1}, & \text{in Type } C_n, \\ q^{2n-1}, & \text{in Type } D_n. \end{cases}$$

(d) If $\mathfrak{g}$ is type B_n, C_n or D_n, and $M = L(\mu)$ where μ is a dominant integral weight then Φ_k is a representation of the cyclotomic BMW algebra $\tilde{\mathcal{Z}}_{r,1,k}$ with q and z as in (c),

$$Q_r = \sum_{\mu^\pm} q^{r\tilde{c}(\mu^\pm,\mu)} \frac{\dim_q(L(\mu^\pm))}{\dim_q(L(\mu))\dim_q(L(\omega_1))}, \qquad r \in \mathbb{Z}_{>0},$$

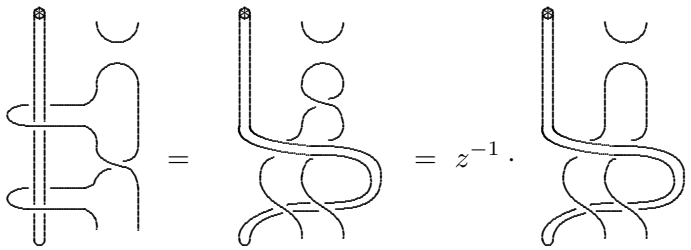
and any (multi)set of parameters $u_1, \dots, u_r$ containing the (multi)set of values $q^{\tilde{c}(\mu^\pm,\mu)}$ as $\mu^\pm$ runs over the dominant integral weights appearing in the decomposition (6.14) of $L(\mu)\otimes L(\omega_1)$. Here

$$\tilde{c}(\mu^\pm, \mu) = \begin{cases} -y, & \text{if } \mu^\pm = \mu, \\ 2c(\mu^\pm/\mu), & \text{if } \mu^\pm \supseteq \mu, \\ -2(c(\mu/\mu^\pm) + y), & \text{if } \mu^\pm \subseteq \mu, \end{cases}$$

where y and $c(b)$ are as defined in (6.13) and (6.15), respectively.

Proof. (a) It is only necessary to show that $\Phi_k(T_i) = q^{1/(n+1)}\check{R}_i$ satisfies $(q^{1/(n+1)}\check{R}_i)^2 = (q - q^{-1})(q^{1/(n+1)}\check{R}_i) + 1$ for $2 \leq i \leq n$. This is proved in [LR, Prop. 4.4].

(c) The arguments establishing the relations (6.3a-c) in the definition of the affine BMW algebra are exactly as in [LR, Prop. 5.10]. It remains to establish (6.3d-e). The element E_1 in the affine BMW algebra acts on $V^{\otimes 2}$ as $x \cdot \mathrm{pr}_0$ where pr_0 is the unique $U_h\mathfrak{g}$-invariant projection onto the invariants in $V^{\otimes 2}$ and x is as in (6.5). Using the identity (6.9) and the pictorial equalities



it follows that $\Phi_2(E_1X^{\varepsilon_1}T_1X^{\varepsilon_1})$ acts as $xz^{-1}\cdot\check{R}_{L(0),M}\check{R}_{M,L(0)}(\mathrm{id}_M\otimes\mathrm{pr}_0)$. By (2.11), this is equal to

$$z^{-1}\cdot(C_M\otimes C_{L(0)})C^{-1}_{M\otimes L(0)}\Phi_2(\mathrm{id}_M\otimes E_1)=z^{-1}\cdot C_MC_M^{-1}\Phi_2(\mathrm{id}_M\otimes E_1)=z^{-1}\cdot\Phi_2(E_1),$$

establishing the relation in (6.3e).

Since $\Phi_2(E_1)$ acts as $x\cdot(\mathrm{id}_M\otimes\mathrm{pr}_0)$ on $M\otimes V^{\otimes 2}$ the morphism $\Phi_2(E_1X^{r\varepsilon_1}E_1)$ is a morphism from $M\otimes L(0)\to M\otimes L(0)$. Since $M=M\otimes L(0)$ is a highest weight module this morphism is $Q_r\cdot\mathrm{id}_M$, for some $Q_r\in\mathbb{C}$. By the results of Drinfeld [Dr] and Reshetikhin [Re] (see [Ba, p. 250]), the action of the morphism $\Phi_2(E_1X^{r\varepsilon_1}E_1)$ corresponds to the action of a central element of $U_h\mathfrak{g}$ on M. Thus the constant Q_r depends only on the central character of M.

(b) Let $b_1,\dots,b_r$ be the addable boxes of μ and consider the action of X^{ε_1} on $M\otimes V=L(\mu)\otimes L(\omega_1)$. We will show that $\Phi_k(X^{\varepsilon_1})=\check{R}_0^2$ satisfies the relation $(\check{R}_0^2-u_1)\cdots(\check{R}_0^2-u_r)=0$, where $u_i=q^{2c(b_i)}$. By (2.11) and (2.14) it follows that

$$\check{R}_0^2=\sum_{\mu^+}q^{\langle\mu^+,\,\mu^++2\rho\rangle-\langle\mu\,,\,\mu+2\rho\rangle-\langle\omega_1\,,\,\omega_1+2\rho\rangle}P^{\mu^+}_{\mu,\omega_1}=\sum_{\mu^+}q^{2c(\mu^+/\mu)}P^{\mu^+}_{\mu,\omega_1},$$

where the sum is over all partitions μ^+ obtained by adding a box to μ, $P^{\mu^+}_{\mu,\omega_1}$ is the projection onto $L(\mu^+)$ in the tensor product $M\otimes V=L(\mu)\otimes L(\omega_1)$, and $c(\mu^+/\mu)$ is the content of the box μ^+/μ which is added to μ to get μ^+. Thus $\check{R}_0^2$ is a diagonal operator with eigenvalues $q^{2c(\mu^+/\mu)}$ and so it satisfies the equation (6.2).

(d) Using the appropriate case of the decomposition rule for $L(\mu)\otimes L(\omega_1)$, the proof of the relation $(X^{\varepsilon_1}-u_1)\cdots(X^{\varepsilon_1}-u_r)=0$ is as in (b). The values of $\tilde{c}(\mu^{\pm},\mu)$ are determined from (6.16). To compute the value of Q_r note that $\Phi_2(E_1X^{r\varepsilon_1}E_1)=\Phi_2(\varepsilon_1(X^{r\varepsilon_1})E_1)$, in the notations of the proof of Theorem 5.4. Thus Q_r is determined by the formula in Theorem 5.4 and the decomposition of $L(\mu)\otimes L(\omega_1)$ in (6.14). ∎

Remark. The parameters in $Q_1,Q_2,\dots\in\mathbb{C}$ needed in Theorem 6.17c can be determined by using the formula of Baumann [Ba, Theorem 1] which characterizes Q_r in terms of the values Q_1 given in (5.8). To do this it is necessary to use formula (5.8) for Q_1 several times: μ is always the highest weight of M, but many different ν will be needed. Note that the proof of the formula (5.8) for Q_1 in [TW] does not require μ to be dominant integral.

The following theorem provides an analogue of Schur-Weyl duality for the affine Hecke algebras, cyclotomic Hecke algebras, affine BMW algebras and cyclotomic BMW algebras. Alternative Schur-Weyl dualities have been given by Chari-Pressley [CP2] for the case of affine Hecke algebras and by Sakamoto and Shoji [SS] for cyclotomic Hecke algebras. Cherednik [Ch] also used a Schur-Weyl duality for the affine Hecke algebra which is different from the Schur-Weyl duality given here.

Theorem 6.18. *Assume that $\mathfrak{g}$ is not of type D_n. Let μ be a dominant integral weight and let $M=L(\mu)$. In each of the cases given in Theorem 6.17 the representation Φ_k is surjective.*

Proof. Part (a) is a consequence of (b) since the representation of $\tilde{H}_k$ in (a) is the composition of the representation $\Phi_k\colon H_{r,1,k}\to\mathrm{End}_{U_h\mathfrak{g}}(L(\mu)\otimes V^{\otimes k})$ from (b) with the surjective algebra homomorphism $\tilde{H}_k\to H_{r,1,k}$ coming from the definition of $H_{r,1,k}$. Similarly part (c) is a consequence of part (d). The proof of the surjectivity of the representation in Theorem 6.17b and Theorem

6.17d are exactly the same as the proofs of [LR, Cor. 4.15] and [LR, Cor. 5.22], respectively. The case considered there is the $\mu = 0$ case but all the arguments there generalize verbatim to the case when μ is an arbitrary dominant integral weight. In [LR, §4] the elements X^{ε_i} in the affine braid group are denoted D_i. The assumption $n >> k$ in [LR] is uncessary for this theorem if the full decomposition rule given in (6.14) is used.

The main point is that the eigenvalues of $X^{\varepsilon_1}, \dots, X^{\varepsilon_i}$ separate the components of the decomposition of $L(\mu) \otimes V^{\otimes i}$. By induction it is sufficient to check that the eigenvalues of X^{ε_1} distiguish the components of $L(\lambda) \otimes V$ for all λ. By (2.10), (2.11) and (2.14), the eigenvalues of X^{ε_i} are of the form $q^{2\tilde{c}(\lambda^{\pm},\lambda)}$ where λ is a dominant integral weight $\tilde{c}(\lambda^{\pm}, \lambda)$ is as in Theorem 6.17d and $\lambda^{\pm}$ runs over the components in the decomposition (6.14) of $L(\lambda) \otimes V$. Different addable boxes for λ can never have the same content since they cannot be in the same diagonal. Similarly for two different removable boxes. Let b be an addable box and b' a removable box for λ. Unless $\mathfrak{g}$ is type D_n and b and b' are in row n, we have $c(b), c(b') \geq -n - 1$. Thus, when $\mathfrak{g}$ is not of type D_n, $c(b) \neq -c(b') - y$ and so the two eigenvalues coming from these boxes are different. ∎

Let $\tilde{\mathcal{Z}}_k$ denote the affine Hecke algebra, the cyclotomic Hecke algebra, the affine BMW algebra or the cyclotomic BMW algebra corresponding to the case of Theorem 6.17 which is being considered. Then, as in the classical Schur-Weyl duality setting, Theorem 6.18 implies that as $(U_h\mathfrak{g}, \tilde{\mathcal{Z}}_k)$ bimodules

$$L(\mu) \otimes V^{\otimes k} \cong \bigoplus_{\lambda} L(\lambda) \otimes \mathcal{L}^{\lambda/\mu}, \tag{6.19}$$

where $L(\lambda)$ is the irreducible $U_h\mathfrak{g}$-module of highest weight λ and $\mathcal{L}^{\lambda/\mu}$ is the irreducible $\tilde{\mathcal{Z}}_k$ module defined by 4.1.

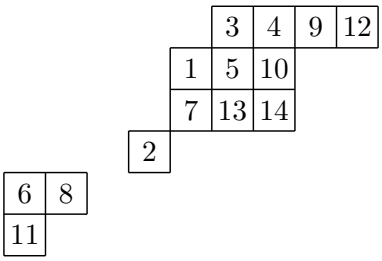
The irreducible $\tilde{\mathcal{Z}}_k$ modules $\mathcal{L}^{\lambda/\mu}$ appearing in (6.19) can be constructed quite explicitly. All the necessary computations for doing this have already been done in [LR, §4 and 5] which does the case $\mu = 0$. All the arguments in [LR, §4 and 5] generalize directly to the case when μ is an arbitrary dominant integral weight. The final result is Theorem 6.20 below. The result in part (a) of Theorem 6.20 is due to Cherednik [Ch].

If λ and μ are partitions such that $\lambda \supseteq \mu$ the *skew shape* λ/μ is the configuration of boxes of in λ which are not in μ. Let λ/μ be a skew shape with k boxes. A *standard tableau* of shape λ/μ is a filling T of the boxes of λ/μ with $1, 2, \dots, k$ such that

(a) the rows of T are increasing (left to right), and

(b) the columns of T are increasing (top to bottom).

For example,



is a standard tableau of shape $\lambda/\mu = (977421)/(5443)$.

For any two partitions μ and λ an *up down tableau of length k from μ to λ* is a sequence of partitions $T = \big(\mu = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k-1)}, \tau^{(k)} = \lambda\big)$ such that

(a) $\tau^{(i)} \supseteq \tau^{(i-1)}$ and $\tau^{(i)}/\tau^{(i-1)} = \square$, or (b) $\tau^{(i-1)} \supseteq \tau^{(i)}$ and $\tau^{(i-1)}/\tau^{(i)} = \square$,

and, in type B_n the situation $\tau^{(i-1)} = \tau^{(i)}$ with $\ell(\tau^{(i-1)}) = n$ is also allowed. Note that a standard tableau λ/μ with k boxes is exactly an up down tableau of length k from μ to λ where all steps in the sequence satisfy condition (a).

Theorem 6.20.

(a) *Let λ/μ be a skew shape with k boxes. Then the module $\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu))$ for the affine Hecke algebra $\tilde{H}_k$ is irreducible and is given by*

$$\mathcal{L}^{\lambda/\mu} = span\{v_T \mid T \text{ standard tableaux of shape } \lambda/\mu\}$$

(so that the symbols v_T are a $\mathbb{C}$-basis of $\mathcal{L}^{\lambda/\mu}$) with $\tilde{H}_k$ action given by

$$\begin{aligned} X^{\varepsilon_i} v_T &= q^{2c(T(i))} v_T, && 1 \le i \le k, \\ T_i v_T &= (T_i)_{TT} v_T + \sqrt{\big(q^{-1} + (T_i)_{TT}\big)\big(q^{-1} + (T_i)_{s_iT,s_iT}\big)}\ v_{s_iT}, && 1 \le i \le k-1, \end{aligned}$$

where

$(T_i)_{TT}$ *is the constant* $\dfrac{q - q^{-1}}{1 - q^{2c(T(i))-c(T(i+1))}}$,

$c(b)$ *denotes the content of the box* b,

$T(i)$ *is the box containing* i *in* T,

s_iT *is the same filling as* T *except* i *and* $i+1$ *are switched, and*

$v_{s_iT} = 0$ *if* s_iT *is not a standard tableau.*

(b) *Let λ/μ be a pair of partitions. Then the module $\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu))$ for the affine BMW algebra $\tilde{\mathcal{Z}}_k$ is irreducible and is given by*

$$\mathcal{L}^{\lambda/\mu} = span\left\{ v_T \;\middle|\; \begin{array}{c} T = \big(\mu = \tau^{(0)}, \ldots, \tau^{(k)} = \lambda\big) \text{ an} \\ \text{up down tableau of length } k \text{ from } \mu \text{ to } \lambda \end{array} \right\}$$

(so that the symbols v_T are a $\mathbb{C}$-basis of $\mathcal{L}^{\lambda/\mu}$) with $\tilde{\mathcal{Z}}_k$ action given by

$$X^{\varepsilon_i} v_T = q^{\tilde{c}(\tau^{(i)}, \tau^{(i-1)})} v_T, \qquad 1 \le i \le k,$$

$$E_i v_T = \delta_{\tau^{(i+1)}, \tau^{(i-1)}} \cdot \sum_S (E_i)_{ST} v_S, \qquad \text{and} \qquad T_i v_T = \sum_S (T_i)_{ST} v_S, \qquad 1 \le i \le k-1,$$

where both sums are over up-down tableaux $S = \big(\mu = \tau^{(0)}, \ldots, \tau^{(i-1)}, \sigma^{(i)}, \tau^{(i+1)}, \ldots, \tau^{(k)} = \lambda\big)$ that are the same as T except possibly at the ith step and

$$(E_i)_{ST} = \epsilon \cdot \frac{\sqrt{\dim_q(L(\tau^{(i)}))\dim_q(L(\sigma^{(i)}))}}{\dim_q(\tau^{(i-1)})},$$

$$(T_i)_{ST} = \begin{cases} \sqrt{\big(q^{-1} + (T_i)_{TT}\big)\big(q^{-1} + (T_i)_{SS}\big)}, & \text{if } \tau^{(i-1)} \ne \tau^{(i+1)} \text{ and } S \ne T, \\ \left(\dfrac{q - q^{-1}}{1 - \tilde{c}(\tau^{(i+1)}, \sigma^{(i)})\tilde{c}(\tau^{(i)}, \tau^{(i-1)})^{-1}}\right)(\delta_{ST} - (E_i)_{ST}), & \text{otherwise,} \end{cases}$$

$$\tilde{c}(\tau^{(i)}, \tau^{(i-1)}) = \begin{cases} z^{-1}, & \text{if } \tau^{(i)} = \tau^{(i-1)}, \\ q^{2c(\tau^{(i)}/\tau^{(i-1)})}, & \text{if } \tau^{(i)} \supseteq \tau^{(i-1)}, \\ z^{-2} q^{-2c(\tau^{(i-1)}/\tau^{(i)})}, & \text{if } \tau^{(i)} \subseteq \tau^{(i-1)}, \end{cases}$$

and $\epsilon = 1$, in type B_n and D_n, and $\epsilon = -1$ in type C_n.

Markov traces on affine and cyclotomic Hecke and BMW algebras

If $M = L(\mu)$ where μ is a dominant integral weight and $V = L(\omega_1)$ then each of the representations $\Phi_k \colon \tilde{\mathcal{Z}}_k \to \mathrm{End}_{U_h\mathfrak{g}}(M \otimes V^{\otimes k})$ (where $\tilde{\mathcal{Z}}_k$ is the affine Hecke algebra, the cyclotomic Hecke algebra, the affine BMW algebra, or the cyclotomic BMW algebra) gives rise to a Markov trace via Theorem 5.4. The parameters and the weights of these Markov traces are given by Theorems 5.4 and 5.11.

In type A case λ/μ is a skew shape with k boxes and the parameters and the weights of (most of) these traces have been given in terms of partitions in [GIM]. In [GIM], μ is a partition of a special form ([GIM, 2.2(*)]) and so, in their case, the skew shape λ/μ can be viewed as an r-tuple of partitions. Their formulas can be recovered from ours by rewriting the quantum dimension $\dim_q(L(\lambda))$ from (5.3) in terms of the partition as in [Mac, I §3 Ex. 1]:

$$\dim_q(L(\lambda)) = \prod_{b\in\lambda} \frac{[n+1+c(b)]}{[h(b)]}, \qquad \text{in the type } A_n \text{ case,} \tag{6.21}$$

where, if b is the box in position (i,j) of λ, then $h(b) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook length* at b, and $[d] = (q^d - q^{-d})/(q - q^{-1})$ for a positive integer d. Thus, the first formula in [GIM, §2.3] coincides with $\dim_q(L(\lambda))/(\dim_q(V))^{|\lambda|}$ and so the formula for the weights of the Markov trace on cyclotomic Hecke algebras which is given in [GIM, Prop. 2.3] coincides exactly with the formula in Theorem 5.11. To remove the constants that come from the difference between $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ the affine braid group action in Theorem 6.17a should be normalized so that $\Phi_k(X^{\varepsilon_1}) = q^{2|\mu|/(n+1)}\check{R}_0^2$ and $\Phi_k(T_i) = q^{1/(n+1)}\check{R}_i$. Then, from Theorem 5.4, (6.14) and (6.16) it follows that the parameters of the Markov trace are $z = q^{n+1}/[n+1]$ and

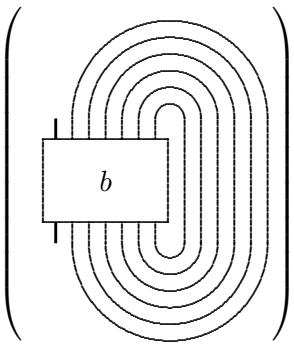
$$\begin{aligned}
Q_r &= \sum_{\mu^+} q^{2rc(\mu^+/\mu)} \frac{\dim_q(L(\mu^+))}{\dim_q(L(\mu))\dim_q(V)} \\
&= \sum_{\mu^+} q^{2rc(\mu^+/\mu)} \left(\prod_{b\in\mu^+} \frac{[n+1+c(b)]}{[h(b)]}\right)\left(\prod_{b\in\mu} \frac{[h(b)]}{[n+1+c(b)]}\right)\frac{1}{[n+1]} \\
&= \sum_{\mu^+} q^{2rc(\mu^+/\mu)} \left(\frac{\prod_{b\in\mu}[h(b)]}{\prod_{b\in\mu^+}[h(b)]}\right)\frac{[n+1+c(\mu^+/\mu)]}{[n+1]} \\
&= \sum_{\mu^+} q^{2rc(\mu^+/\mu)} \left(\prod_{b'} \frac{[h(b')]}{[h(b')+1]}\right)\left(\prod_{b''} \frac{[h(b'')]}{[h(b'')+1]}\right)\frac{[n+1+c(\mu^+/\mu)]}{[n+1]},
\end{aligned}$$

where, in the last expression, the first product is over boxes $b' \in \mu$ which are in the same row as the added box μ^+/μ and the second product is over $b'' \in \mu$ which are in the same column as μ^+/μ. Then cancellation of the common terms in the numerator and denominators of each product yields the combinatorial formulas for the parameters of the Markov traces on cyclotomic Hecke algebras which are given in [GIM, Thm. 2.4].

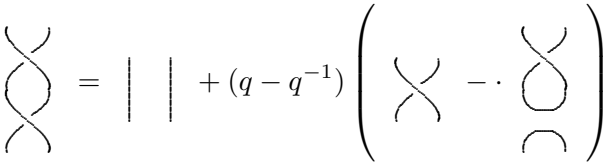
Lambropoulou [Lb, §4] has proved that there is a *unique* Markov trace on the affine Hecke algebra with a given choice of parameters $z, Q_1, \ldots, Q_r \in \mathbb{C}$. A similar result is true for the affine BMW algebra.

Theorem 6.22. *For each fixed choice of parameters q, z and $Q_1, Q_2, \ldots$ there is a unique Markov trace on the affine BMW algebra $\mathcal{Z}_k$.*

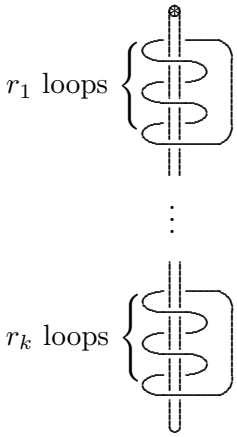
Sketch of proof. Consider the image of an affine braid b in the affine BMW algebra. The Markov trace of this braid can be viewed pictorially as the closure of the braid b.

$\mathrm{mt}_k(b) = \mathrm{mt}_1$ 

Consider a string in the closure as it winds around the other strings and the pole. If the string crosses another string twice without going around the pole between these two crossings then we can use the relation



to rewrite the closed braid as a linear combination of closed braids with *fewer* crossings between strings. By successive steps of this type we can reduce the computation of the Markov trace of a braid to a linear combination of



$$= \frac{Q_{r_1}\cdots Q_{r_k}}{\dim_q(V)^k} \cdot \mathrm{mt}\left(\right) = \frac{Q_{r_1}\cdots Q_{r_k}}{\dim_q(V)^k}. \qquad \blacksquare$$

Remark. For computations it is helpful to note that

$$\dim_q(L(\omega_1)) = \begin{cases} [n+1], & \text{in type } A_n, \\ [2n]+1, & \text{in type } B_n, \\ [2n+1]-1, & \text{in type } C_n, \\ [2n-1]+1, & \text{in type } D_n. \end{cases} \tag{6.23}$$

Standard and simple modules for affine Hecke algebras

The original construction of the irreducible representations of the affine Hecke algebra of type A is due to Zelevinsky [Ze2] and is an analogue of the Langlands construction of admissible representations of real reductive Lie groups. Zelevinsky used the combinatorics of multisegments which is easily seen to be equivalent to the combinatorics of unipotent-semisimple pairs used later in [KL] (see [Ar]). Here we show how the construction of affine Hecke algebra representations via the functors F_λ naturally matches up with Zelevinsky's indexings by multisegments. Using the multisegment indexing of representations, Theorem 6.31 below explicitly matches up the decomposition numbers for affine Hecke algebras with Kazhdan-Lusztig polynomials. Recall that the functor F_λ gives representations of the affine Hecke algebra in the setting of Theorem 6.17a when $\mathfrak{g}$ is of type A_n and $V = L(\omega_1)$ is the n-dimensional fundamental representation.

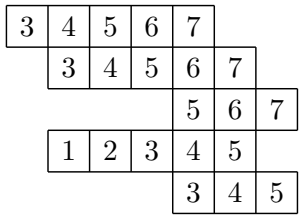
Consider an (infinite) sheet of graph paper which has its diagonals labeled consecutively by $\ldots, -2, -1, 0, 1, 2, \ldots$. The *content* $c(b)$ of a box b on this sheet of graph paper is

$$c(b) = \text{the diagonal number of the box } b$$

(a natural generalization of the definition of $c(b)$ in (6.15)). A *multisegment* is a collection of rows of boxes (segments) placed on graph paper. We can label this multisegment by a pair of weights $\lambda = \lambda_1\varepsilon_1 + \cdots \lambda_{n+1}\varepsilon_{n+1}$ and $\mu = \mu_1\varepsilon_1 + \cdots + \mu_{n+1}\varepsilon_{n+1}$ by setting

$$\begin{aligned}(\lambda+\rho)_i &= \text{content of the last box in row } i, \qquad \text{and}\\ (\mu+\rho)_i &= (\text{content of the first box in row } i) - 1.\end{aligned}$$

For example



corresponds to $\lambda+\rho = (7,\ 7,\ 7,\ 5,\ 5)$ and $\mu+\rho = (2,\ 2,\ 4,\ 0,\ 2)$ (6.24)

(the numbers in the boxes in the picture are the contents of the boxes). The construction forces the condition

(a) $(\lambda+\rho)_i - (\mu+\rho)_i \in \mathbb{Z}_{\geq 0}$.

and since we want to consider *unordered* collections of boxes it is natural to take the following pseudo-lexicographic ordering on the segments

(b) $(\lambda+\rho)_i \geq (\lambda+\rho)_{i+1}$,

(c) $(\mu+\rho)_i \leq (\mu+\rho)_{i+1}$ if $(\lambda+\rho)_i = (\lambda+\rho)_{i+1}$,

when we denote the multisegment λ/μ by a pair of weights λ, μ. In terms of weights the conditions (a), (b) and (c) can be restated as (note that in this case both λ and μ are integral)

(a′) $\lambda - \mu$ is a weight of $V^{\otimes k}$, where k is the number of boxes in λ/μ,

(b′) λ is integrally dominant,

(c′) $\mu = w \circ \nu$ with ν integrally dominant and w maximal length in the coset $W_{\lambda+\rho} w W_{\nu+\rho}$,

These conditions on the pair of weights (λ, μ) arose previously in Proposition 4.3d and Lemma 4.7.

Let λ/μ be a multisegment with k boxes and number the boxes of λ/μ from left to right (like a book). Define

$$\tilde{H}_{\lambda/\mu} = \text{ subalgebra of } \tilde{H}_k \text{ generated by } \{X^\lambda,\ T_j \mid \lambda \in L,\ \mathrm{box}_j \text{ is not at the end of its row}\},$$

so that $\tilde{H}_{\lambda/\mu}$ is the "parabolic" subalgebra of $\tilde{H}_k$ corresponding to the multisegment λ/μ. Define a one-dimensional $\tilde{H}_{\lambda/\mu}$ module $\mathbb{C}_{\lambda/\mu} = \mathbb{C}v_{\lambda/\mu}$ by setting

$$X^{\varepsilon_i} v_{\lambda/\mu} = q^{2c(\mathrm{box}_i)} v_{\lambda/\mu}, \qquad \text{and} \qquad T_j v_{\lambda/\mu} = q v_{\lambda/\mu}, \tag{6.25}$$

for $1 \le i \le k$ and j such that box_j is not at the end of its row.

Let $\mathfrak{g}$ be of type A_n and let F_λ be the functor $\mathrm{Hom}_{U_h\mathfrak{g}}(M(\lambda), \cdot \otimes V^{\otimes k})$ from the setting of Theorem 6.17a, where $V = L(\omega_1)$. The *standard module* for the affine Hecke algebra $\tilde{H}_k$ is

$$\mathcal{M}^{\lambda/\mu} = F_\lambda(M(\mu)) \tag{6.26}$$

as defined in (4.1). It follows from the above discussion that these modules are naturally indexed by multisegments λ/μ. The following proposition shows that this standard module coincides with the usual standard module for the affine Hecke algebra as considered by Zelevinsky [Ze2] (see also [Ar], [CG] and [KL]).

Proposition 6.27. *Let λ/μ be a multisegment determined by a pair weights (λ, μ) with λ integrally dominant. Let $\mathbb{C}_{\lambda/\mu}$ be the one dimensional representation of the parabolic subalgebra $\tilde{H}_{\lambda/\mu}$ of the affine Hecke algebra $\tilde{H}_k$ defined in (6.25). Then*

$$\mathcal{M}^{\lambda/\mu} \cong Ind_{\tilde{H}_{\lambda/\mu}}^{\tilde{H}_k}(\mathbb{C}_{\lambda/\mu}).$$

Proof. To remove the constants that come from the difference between $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ the affine braid group action in Theorem 6.17a should be normalized so that $\Phi_k(X^{\varepsilon_1}) = q^{2|\mu|/(n+1)}\check{R}_0^2$ and $\Phi_k(T_i) = q^{1/(n+1)}\check{R}_i$.

By Proposition 4.3a, $\mathcal{M}^{\lambda/\mu} \cong (V^{\otimes k})_{\lambda-\mu}$ as a vector space. Let $\{v_1, v_2, \ldots, v_{n+1}\}$ be the standard basis of $V = L(\omega_1)$ with $\mathrm{wt}(v_i) = \varepsilon_i$. If we let the symmetric group S_k act on $V^{\otimes k}$ by permuting the tensor factors then

$$(V^{\otimes k})_{\lambda-\mu} = \text{span-}\{\pi \cdot v^{\otimes(\lambda-\mu)} \mid \pi \in S_k\} = \text{span-}\{\pi \cdot v^{\otimes(\lambda-\mu)} \mid \pi \in S_k/S_{\lambda-\mu}\}, \qquad \text{where}$$

$$v^{\otimes(\lambda-\mu)} = \underbrace{v_1 \otimes \cdots \otimes v_1}_{\lambda_1-\mu_1} \otimes \cdots \otimes \underbrace{v_n \otimes \cdots \otimes v_n}_{\lambda_n-\mu_n} \qquad \text{and} \qquad S_{\lambda-\mu} = S_{\lambda_1-\mu_1} \times \cdots \times S_{\lambda_n-\mu_n}$$

is the parabolic subgroup of S_k which stabilizes the vector $v^{\otimes(\lambda-\mu)} \in V^{\otimes k}$. This shows that, as vector spaces,

$$\mathcal{M}^{\lambda/\mu} \cong \mathrm{Ind}_{\tilde{H}_{\lambda/\mu}}^{\tilde{H}_k}(\mathbb{C}_{\lambda/\mu}) = \text{span-}\{T_\pi \otimes v_{\lambda/\mu} \mid \pi \in S_k/S_{\lambda-\mu}\} \tag{6.28}$$

are isomorphic.

For notational purposes let

$$b_{\lambda/\mu} = v_\mu^+ \otimes v^{\otimes(\lambda-\mu)} = v_\mu^+ \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}$$

and let $\bar{b}_{\lambda/\mu}$ be the image of $b_{\lambda/\mu}$ in $(M \otimes V^{\otimes k})^{[\lambda]}$. Since λ is integrally dominant and $\bar{b}_{\lambda/\mu}$ has weight λ it must be a highest weight vector. We will show that X^{ε_ℓ} acts on $\bar{b}_{\lambda/\mu}$ by the constant $q^{c(\mathrm{box}_\ell)}$, where $c(\mathrm{box}_\ell)$ is the content of the ℓth box of the multisegment λ/μ (read left to right and top to bottom like a book).

Consider the projections

$$pr_\ell \colon M(\mu) \otimes V^{\otimes k} \to (M(\mu) \otimes V^{\otimes \ell})^{[\lambda^{(\ell)}]} \otimes V^{\otimes(k-\ell)} \qquad \text{where} \qquad \lambda^{(\ell)} = \mu + \sum_{j \le \ell} \mathrm{wt}(v_{i_\ell})$$

and pr_i acts as the identity on the last $k-i$ factors of $M(\mu) \otimes V^{\otimes k}$. Then

$$\bar{b}_{\lambda/\mu} = pr_k pr_{k-1} \cdots pr_1 b_{\lambda/\mu},$$

and for each $1 \le \ell \le k$, (the first ℓ components of) $pr_{\ell-1} \cdots pr_1(b_{\lambda/\mu})$ form a highest weight vector of weight $\lambda^{(\ell)}$ in $M \otimes V^{\otimes \ell}$. It is the "highest" highest weight vector of

$$((M(\mu) \otimes V^{\otimes(\ell-1)})^{[\lambda^{(\ell-1)}]} \otimes V)^{[\lambda^{(\ell)}]} \tag{6.29}$$

with respect to the ordering in Lemma 4.2 and thus it is deepest in the filtration constructed there. Note that the quantum Casimir element acts on the space in (6.29) as the constant $q^{\langle \lambda^{(\ell)}, \lambda^{(\ell)}+2\rho\rangle}$ times a unipotent transformation, and the unipotent transformation must preserve the filtration coming from Lemma 4.2. Since $pr_\ell(b_{\lambda/\mu})$ is the highest weight vector of the smallest submodule of this filtration (which is isomorphic to a Verma module by Lemma 4.2b) it is an eigenvector for the action of the quantum Casimir. Thus, by (2.11) and (2.13), X^{ε_ℓ} acts on $pr_\ell(b_{\lambda/\mu})$ by the constant

$$q^{\langle \lambda^{(\ell)}, \lambda^{(\ell)}+2\rho\rangle - \langle \lambda^{(\ell-1)}, \lambda^{(\ell-1)}+2\rho\rangle - \langle \omega_1, \omega_1+2\rho\rangle} = q^{2c(\mathrm{box}_\ell)}.$$

(see [LR] Since X^{ε_ℓ} commutes with pr_j for $j > \ell$ it this also specifies the action of X^{ε_ℓ} on $\bar{b}_{\lambda/\mu} = pr_\ell(b_{\lambda/\mu})$.

The explicit R-matrix $\check{R}_{VV} \colon V \otimes V \to V \otimes V$ for this case ($\mathfrak{g}$ of type A and $V = L(\omega_1)$) is well known (see, for example, the proof of [LR, Prop. 4.4]) and given by

$$(v_i \otimes v_j) q^{1/(n+1)} \check{R}_{VV} = \begin{cases} v_j \otimes v_i, & \text{if } i > j, \\ (q - q^{-1}) v_i \otimes v_j + v_j \otimes v_i, & \text{if } i < j, \\ q v_i \otimes v_j, & \text{if } i = j. \end{cases}$$

Since T_i acts by $\check{R}_{VV}$ on the ith and $(i+1)$st tensor factors of $V^{\otimes k}$ and commutes with the projection pr_λ it follows that $T_j(\bar{b}_{\lambda/\mu}) = q\,\bar{b}_{\lambda/\mu}$, if box_j is not a box at the end of a row of λ/μ. This analysis of the action of $\tilde{H}_{\lambda/\mu}$ on $\bar{b}_{\lambda/\mu}$ shows that there is an $\tilde{H}_k$-homomorphism

$$\begin{array}{ccc} \mathrm{Ind}_{\tilde{H}_{\lambda/\mu}}^{\tilde{H}_k}(\mathbb{C} v_{\lambda/\mu}) & \longrightarrow & \mathcal{M}^{\lambda/\mu} \\ v_{\lambda/\mu} & \longmapsto & \bar{b}_{\lambda/\mu}. \end{array}$$

This map is surjective since $\mathcal{M}^{\lambda/\mu}$ is generated by $\bar{b}_{\lambda/\mu}$ (the $\mathcal{B}_k$ action on $v^{\lambda-\mu}$ generates all of $(V^{\otimes k})_{\lambda-\mu}$). Finally, (6.28) guarantees that it is an isomorphism. ∎

In the same way that each weight $\mu \in \mathfrak{h}^*$ has a *normal form*

$$\mu = w \circ \tilde{\mu}, \qquad \text{with} \qquad \begin{array}{l} \tilde{\mu} \text{ integrally dominant, and} \\ w \text{ maximal length in the coset } wW_{\tilde{\mu}+\rho}, \end{array}$$

every multisegment λ/μ has a *normal form*

$$\lambda/\mu = \nu/(w \circ \tilde{\nu}), \qquad \text{with} \qquad \begin{array}{l} \nu+\rho \text{ the sequence of contents of boxes of } \lambda/\mu, \\ \tilde{\nu} = \nu - (1,1,\dots,1), \quad \text{and} \\ w \text{ maximal length in } W_{\nu+\rho} w W_{\nu+\rho}. \end{array}$$

The element w in the normal form $\nu/(w \circ \tilde{\nu})$ of λ/μ can be constructed combinatorially by the following scheme. We number (order) the boxes of λ/μ in two different ways.

First ordering: To each box b of λ/μ associate the following triple

$$\big(\text{content of the box to the left of } b, -(\text{content of } b), -(\text{row number of } b)\big)$$

where, if a box is the leftmost box in a row "the box to its left" is the rightmost box in the same row. The lexicographic ordering on these triples induces an ordering on the boxes of λ/μ.

Second ordering: To each box b of λ/μ associate the following pair

$$\big(\text{content of } b, -(\text{the number of box } b \text{ in the first ordering})\big)$$

The lexicographic ordering of these pairs induces a second ordering on the boxes of λ/μ.

If v is the permutation defined by these two numberings of the boxes then $w = w_0 v w_0$. For example, for the multisegment λ/μ displayed in (6.24) the numberings of the boxes are given by

21	6	10	13	18		
	20	5	9	12	17	
				19	11	16
	15	1	2	4	8	
				14	3	7

first ordering of boxes

and

3	7	12	16	19		
	4	8	13	17	20	
				11	18	21
	1	2	6	9	14	
				5	10	15

second ordering of boxes

and the normal form of λ/μ is

$\nu = (7,7,7,6,6,6,5,5,5,5,5,4,4,4,4,3,3,3,3,2,1)$,

$\tilde{\nu} = (6,6,6,5,5,5,4,4,4,4,4,3,3,3,3,2,2,2,2,1,0)$, and $w = w_0 v w_0$ where

$$v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 15 & 1 & 21 & 20 & 14 & 2 & 6 & 5 & 4 & 3 & 19 & 10 & 9 & 8 & 7 & 13 & 12 & 11 & 18 & 17 & 16 \end{pmatrix}$$

Let $\mathfrak{g}$ be of type A_n and $V = L(\omega_1)$ and let

$$\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu)), \tag{6.30}$$

as defined in (4.1). It is known (a consequence of Proposition 6.27 and Proposition 4.3c) that $\mathcal{L}^{\lambda/\mu}$ is always a simple $\tilde{H}_k$-module or 0. Furthermore, all simple $\tilde{H}_k$ modules are obtained by this construction. See [Su] for proofs of these statements. The following theorem is a reformulation of Proposition 4.12 in terms of the combinatorics of our present setting.

Theorem 6.31. *Let λ/μ and ρ/τ be multisegments with k boxes (with μ and τ assumed to be integral) and let*

$$\lambda/\mu = \nu/(w \circ \tilde{\nu}) \qquad \text{and} \qquad \rho/\tau = \gamma/(v \circ \tilde{\gamma})$$

be their normal forms. Then the multiplicities of $\mathcal{L}^{\rho/\tau}$ in a Jantzen filtration of $\mathcal{M}^{\lambda/\mu}$ are given by

$$\sum_{j\geq 0}\left[\frac{(\mathcal{M}^{\lambda/\mu})^{(j)}}{(\mathcal{M}^{\lambda/\mu})^{(j+1)}}:\mathcal{L}^{\rho/\tau}\right]\mathtt{v}^{\frac{1}{2}(\ell(y)-\ell(w)+j)}=\begin{cases}P_{wv}(\mathtt{v}), & \text{if } \nu=\gamma,\\ 0, & \text{if } \nu\neq\gamma,\end{cases}$$

where $P_{wv}(\mathtt{v})$ is the Kazhdan-Lusztig polynomial for the symmetric group S_k.

Theorem 6.31 says that every decomposition number for affine Hecke algebra representations is a Kazhdan-Lusztig polynomial. The following is a converse statement which says that every Kazhdan-Lusztig polynomial for the symmetric group is a decomposition number for affine Hecke algebra representations. This statement is interesting in that Polo [Po] has shown that *every* polynomial in $1+\mathtt{v}\mathbb{Z}_{\geq 0}[\mathtt{v}]$ is a Kazhdan-Lusztig polynomial for some choice of n and permutations $v,w\in S_n$. Thus, the following proposition also shows that every polynomial arises as a generalized decomposition number for an appropriate pair of affine Hecke algebra modules.

Proposition 6.32. *Let $\lambda=(r,r,\ldots,r)=(r^r)$ and $\mu=(0,0,\ldots,0)=(0^r)$. Then, for each pair of permutations $v,w\in S_r$, the Kazhdan-Lusztig polynomial $P_{vw}(\mathtt{v})$ for the symmetric group S_r is equal to*

$$P_{vw}(\mathtt{v})=\sum_{j\geq 0}\left[\frac{(\mathcal{M}^{\lambda/w\circ\mu})^{(j)}}{(\mathcal{M}^{\lambda/w\circ\mu})^{(j+1)}}:\mathcal{L}^{\lambda/v\circ\mu}\right]\mathtt{v}^{\frac{1}{2}(\ell(y)-\ell(w)+j)}.$$

Proof. Since $\mu+\rho$ and $\lambda+\rho$ are both regular, $W_{\lambda+\rho}=W_{\mu+\rho}=1$ and the standard and irreducible modules $\mathcal{L}^{\lambda/(w\circ\mu)}$ and $\mathcal{M}^{\lambda/(v\circ\mu)}$ ranging over all $v,w\in S_k$. Thus, this statement is a corollary of Proposition 4.12. ∎

7. References

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