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Dissertation, Chapter 1

Representation Theory

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Abstract

In my work on the Brauer algebra, which is not a group algebra but is a semisimple algebra with a distinguished basis, I have used the group algebra of the symmetric group as a guide, and tried to find generalizations to the Brauer algebra of as many of the properties of the symmetric group as possible. One of the outcomes of this work was the discovery that much of the representation theory of general semisimple algebras can be obtained in a fashion exactly analogous to the method used for finite groups. In this chapter I develop this theory from scratch. Along the way I prove, in the setting of semisimple algebras over $\mathbb{C}$, an analogue of Maschke's theorem, a Fourier inversion formula, analogues of the orthogonality relations for characters and a formula giving the character of an induced representation, induced from a semisimple subalgebra. Section 4 reviews the double centralizer theory of I. Schur and defines a "Frobenius map" in the most general setting, a representation of a semisimple algebra. Such a map has proved useful in the study of the characters of the symmetric group, the Brauer algebra, and the Hecke algebra.

1. Representations

An *algebra* is a vector space A over $\mathbb{C}$ with a multiplication such that A is a ring with identity and such that for all $a_1, a_2 \in A$ and $c \in \mathbb{C}$,

$$(ca_1)a_2 = a_1(ca_2) = c(a_1a_2). \tag{1.1}$$

More precisely, an algebra is a vector space over $\mathbb{C}$ with a multiplication that is associative, distributive, has an identity, and satisfies (1.1). Suppose that $a_1, a_2, \ldots, a_n$ is a basis of A and that c_{ij}^k are constants in $\mathbb{C}$ such that

$$a_ia_j = \sum_{k=1}^{n} c_{ij}^k a_k. \tag{1.2}$$

It follows from (1.1) and the distributive property that the equations (1.2) for $1 \le i, j \le n$ completely determine the multiplication in A. The c_{ij}^k are called *structure constants*. The *center* of an algebra A is the subalgebra

$$Z(A) = \{b \in A | ab = ba \text{ for all } a \in A\}.$$

A nonzero element $p \in A$ such that $pp = p$ is called an *idempotent*. Two idempotents $p_1, p_2 \in A$ are orthogonal if $p_1p_2 = p_2p_1 = 0$. A minimal idempotent is an idempotent $p \in A$ that cannot be written as a sum $p = p_1 + p_2$ of orthogonal idempotents $p_1, p_2 \in A$.

For each positive integer d we denote the algebra of $d \times d$ matrices with entries from $\mathbb{C}$ and ordinary matrix multiplication by $M_d(\mathbb{C})$. We denote the $d \times d$ identity matrix in $M_d(\mathbb{C})$ by I_d. For a general algebra A, $M_d(A)$ denotes $d \times d$ matrices with entries in A. We denote the algebra of matrices of the form

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix}, \quad a \in A,$$

by $I_n(A)$. Note that $I_n(A) \cong A$, as algebras. The *trace*, $tr(a)$, of a matrix $a = \|a_{ij}\|$ is the sum of the diagonal entries of a, $tr(a) = \sum_i a_{ii}$.

An *algebra homomorphism* of an algebra A into an algebra B is a $\mathbb{C}$-linear map $f: A \to B$ such that for all $a_1, a_2 \in A$,

$$\begin{aligned} f(1) &= 1, \\ f(a_1a_2) &= f(a_1)f(a_2). \end{aligned} \tag{1.3}$$

A *representation* of an algebra A is an algebra homomorphism

$$V: A \to M_d(\mathbb{C}).$$

The *dimension* of the representation V is d. The image $V(A)$ of the representation V is a finite dimensional algebra of $d \times d$ matrices which we call the *algebra of the representation* V. It is a subalgebra of $M_d(\mathbb{C})$. A *faithful representation* is a representation which is injective. In this case the algebra $V(A)$ is called a faithful realization of A and $A \cong V(A)$. The *character* of the representation V of A is the function $\chi_V: A \to \mathbb{C}$ given by

$$\chi_V(a) = tr(V(a)). \tag{1.4}$$

An *anti-representation* of an algebra A is a $\mathbb{C}$-linear map $V': A \to M_d(\mathbb{C})$ such that for all $a_1, a_2 \in A$,

$$\begin{aligned} V'(1) &= I_d, \\ V'(a_1a_2) &= V'(a_2)V'(a_1). \end{aligned}$$

As before the dimension of the anti-representation is d and the image, $V'(A)$, of the anti-representation is an algebra of matrices called the *algebra of the anti-representation*.

The *group algebra* $\mathbb{C}G$ of a group G is the algebra of formal finite linear combinations of elements of G where the multiplication is given by the linear extension of the multiplication in G. The elements of G constitute a basis of $\mathbb{C}G$. A representation of the group G is a representation of its group algebra.

Let A be an algebra. An *A-module* is a vector space V with an A action $A \times V \to V$ such that for all $a, a_1, a_2 \in A$, $v, v_1, v_2 \in V$, and $c_1, c_2 \in \mathbb{C}$,

$$\begin{aligned} 1v &= v, \\ a_1(a_2v) &= (a_1a_2)v, \\ (a_1+a_2)v &= a_1v + a_2v, \\ a(c_1v_1+c_2v_2) &= c_1(av_1)+c_2(av_2). \end{aligned} \tag{1.5}$$

An *A-module homomorphism* is a $\mathbb{C}$-linear map $f\colon V \to V'$ between A-modules V and V' such that for all $a \in A$ and $v \in V$,

$$f(av) = af(v). \tag{1.6}$$

An *A-module isomorphism* is a bijective A-module homomorphism.

By condition 3 of (1.5) the action of $a \in A$ on V is a linear transformation $V(a)$ of V. If we specify a basis B of V then the linear transformation $V(a)$ can be written as a $d \times d$ matrix, where $\dim V = d$. In this way we associate to every element of A a $d \times d$ matrix. This gives a representation of A which we shall also denote by V.

Conversely, if T is a d dimensional representation of A and V is a d dimensional vector space with basis B then we can define the action of an element a in A by the action of the linear transformation on V determined by the matrix $T(a)$ so that for all $v \in V$,

$$av = T(a)v.$$

In this way V becomes an A-module. Thus the notion of A-module is equivalent to the notion of representation. When we view the A-module we are focusing on the vector space and when we view the representation we are focusing on the linear transformations (matrices).

Let V be an A-module with basis B and let B' be another basis of V and denote the change of basis matrix by P. Let $a \in A$ and let $V(a), V'(a)$ be the matrices, with respect to the bases B and B' respectively, of the linear transformation on V induced by a. Then by elementary linear algebra we have that

$$V'(a) = PV(a)P^{-1}. \tag{1.7}$$

This leads us to the following definition. Two d dimensional representations V and V' of an algebra A are *equivalent* if there exists an invertible $d \times d$ matrix P such that (1.7) holds for all $a \in A$. Isomorphic modules define equivalent representations.

The *direct sum* $V_1 \oplus V_2$ of two A-modules V_1 and V_2 is the A-module of all pairs (v_1, v_2), $v_1 \in V_1$ and $v_2 \in V_2$, with the A action given by

$$a(v_1, v_2) = (av_1, av_2),$$

for all $a \in A$. The *direct sum* $V_1 \oplus V_2$ of two representations V_1 and V_2 of A is the representation V of A given by

$$V(a) = \begin{pmatrix} V_1(a) & 0 \\ 0 & V_2(a) \end{pmatrix}. \tag{1.8}$$

Direct sums of $n > 2$ representations or A-modules are defined analogously. We denote $V \oplus V \oplus \cdots \oplus V$, n factors, by $V^{\oplus n}$. Note that the algebra of the representation $V^{\oplus n}$, $V^{\oplus n}(A)$, is $I_n(V(A))$.

An *A-invariant subspace* of an A-module V is a subspace V' of V such that

$$\{av' \mid a \in A, v' \in V'\} = AV' \subseteq V'.$$

An A-invariant subspace of V is just a submodule of V. Note that the intersection $V' \cap V''$ of any two invariant subspaces V', V'' of V is also an invariant subspace of V.

An A-module with no submodules is a *simple* module. An *irreducible representation* is a representation that is not equivalent to a representation of the form

$$V(a) = \begin{pmatrix} V'(a) & * \\ 0 & * \end{pmatrix}, \tag{1.9}$$

where V' is also representation of A. If V', V'' are invariant subspaces of a representation V and V' is irreducible then $V' \cap V''$ is either equal to 0 or V'. A *completely decomposable* representation is a representation that is equivalent to a direct sum of irreducible representations. An algebra A is called completely decomposable if every representation of A is completely decomposable.

The *centralizer* of an algebra A of $d \times d$ matrices is the algebra $\overline{A}$ of $d \times d$ matrices $\overline{a}$ such that for all matrices $a \in A$,

$$\overline{a}a = a\overline{a}. \tag{1.10}$$

The centralizer of a representation V of an algebra A is the algebra $\overline{V(A)}$.

Examples.

1. Let A be an algebra of $d \times d$ matrices. Since all matrices in A commute with all elements of $\overline{A}$,

$$A \subseteq \overline{\overline{A}}.$$

Also,

$$\overline{I_n(A)} = M_n(\overline{A}) \quad \text{and}$$
$$\overline{M_n(A)} = I_n(\overline{A}).$$

Hence,

$$\overline{\overline{I_n(A)}} = I_n(\overline{\overline{A}}).$$

2. **Schur's lemma.** Let W_1 and W_2 be irreducible representations of A of dimensions d_1 and d_2 respectively. If B is a $d_1 \times d_2$ matrix such that

$$W_1(a)B = BW_2(a), \quad \text{for all } a \in A,$$

then either

1) $W_1 \not\cong W_2$ and $B = 0$, or
2) $W_1 \cong W_2$ and if $W_1 = W_2$ then $B = cI_{d_1}$ for some $c \in \mathbb{C}$.

Proof. B determines a linear transformation $B: W_1 \to W_2$. Since $Ba = aB$ for all $a \in A$ we have that

$$B(aw_1) = Baw_1 = aBw_1 = aB(w_1),$$

for all $a \in A$ and $w_1 \in W_1$. Thus B is an A-module homomorphism. $\ker B$ and $\operatorname{im} B$ are submodules of W_1 and W_2 respectively and are therefore either 0 or equal to W_1 or W_2 respectively. If $\ker B = W_1$ or $\operatorname{im} B = 0$ then $B = 0$. In the remaining case B is a bijection, and thus an isomorphism between W_1 and W_2. In this case we have that $d_1 = d_2$. Thus the matrix B is square and invertible.

Now suppose that $W_1 = W_2$ and let c be an eigenvalue of B. Then the matrix $cI_{d_1} - B$ is such that $W_1(a)(cI_{d_1} - B) = (cI_{d_1} - B)W_1(a)$ for all $a \in A$. The argument in the preceding paragraph shows that $cI_{d_1} - B$ is is either invertible or 0. But if c is an eigenvalue of B then $\det(cI_{d_1} - B) = 0$. Thus $cI_{d_1} - B = 0$. □

3. Suppose that V is a completely decomposable representation of an algebra A and that $V \cong \oplus_\lambda W_\lambda^{\oplus m_\lambda}$ where the W_λ are nonisomorphic irreducible representations of A. Schur's lemma shows that the A-homomorphisms from W_λ to V form a vector space

$$\operatorname{Hom}_A(W_\lambda, V) \cong \mathbb{C}^{\oplus m_\lambda}.$$

The *multiplicity* of the irreducible representation W_λ in V is

$$m_\lambda = \dim \operatorname{Hom}_A(W_\lambda, V).$$

4. Suppose that V is a completely decomposable representation of an algebra A and that $V \cong \oplus_\lambda W_\lambda^{\oplus m_\lambda}$ where the W_λ are nonisomorphic irreducible representations of A and let $\dim W_\lambda = d_\lambda$. Then

$$V(A) \cong \oplus_i W_\lambda^{\oplus m_\lambda}(A) = \oplus_\lambda I_{m_\lambda}(W_\lambda(A)) \cong \oplus_\lambda W_\lambda(A).$$

If we view elements of $\oplus_\lambda I_{m_\lambda} W_\lambda(A)$ as block diagonal matrices with m_λ blocks of size $d_\lambda \times d_\lambda$ for each λ, then by using Ex. 1, and Schur's lemma we get that

$$\begin{aligned}\overline{V(A)} \cong \overline{\oplus_\lambda I_{m_\lambda}(W_\lambda(A))} &= \oplus_\lambda M_{m_\lambda}(\overline{W_\lambda(A)}) \\ &= \oplus_\lambda M_{m_\lambda}(I_{d_\lambda}(\mathbb{C})).\end{aligned}$$

5. Let V be an A-module and let p be an idempotent of A. Then pV is a subspace of V and the action of p on V is a projection from V to pV. If $p_1, p_2 \in A$ are orthogonal idempotents of A then p_1V and p_2V are mutually orthogonal subspaces of V, since if $p_1v = p_2v'$ for some v and v' in V then $p_1v = p_1p_1v = p_1p_2v' = 0$. So $V = p_1V \oplus p_2V$.

6. Let p be an idempotent in A and suppose that for every $a \in A$, $pap = kp$ for some constant $k \in \mathbb{C}$. If p is not minimal then $p = p_1 + p_2$, where $p_1, p_2 \in A$ are idempotents such that $p_1 p_2 = p_2 p_1 = 0$. Then $p_1 = pp_1 p = kp$ for some constant $k \in \mathbb{C}$. This implies that $p_1 = p_1 p_1 = kpp_1 = kp_1$, giving that either $k = 1$ or $p_1 = 0$. So p is minimal.

7. Let A be a finite dimensional algebra and suppose that $z \in A$ is an idempotent of A. If z is not minimal then $z = p_1 + p_2$ where p_1 and p_2 are orthogonal idempotents of A. If any idempotent in this sum is not minimal we can decompose it into a sum of orthogonal idempotents. We continue this process until we have decomposed z as a sum of minimal orthogonal idempotents. At any particular stage in this process z is expressed as a sum of orthogonal idempotents, $z = \sum_i p_i$. So $zA = \sum_i p_i A$. None of the spaces $p_i A$ is 0 since $p_i = p_i \cdot 1 \in p_i A$ and the spaces $p_i A$ are all mutually orthogonal. Thus, since zA is finite dimensional it will only take a finite number of steps to decompose z into minimal idempotents. A *partition of unity* is a decomposition of 1 into minimal orthogonal idempotents.

2. Finite dimensional algebras

The *trace*, $tr(a)$, of a $d \times d$ matrix $a = ||a_{ij}||$ is the sum of its diagonal elements, $tr(a) = \sum_i a_{ii}$. A *trace* $\vec{t}$ on an algebra A is a $\mathbb{C}$-linear map $\vec{t}: A \to \mathbb{C}$ such that for all $a, b \in A$,

$$\vec{t}(ab) = \vec{t}(ba). \tag{2.1}$$

Every representation V of A determines a trace $\vec{t}_V$ on A given by $\vec{t}_V(a) = tr(V(a))$ where $a \in A$. A trace $\vec{t}$ is *nondegenerate* if for each $a \in A$, $a \neq 0$, there exists $b \in A$ such that $\vec{t}(ba) \neq 0$. A trace $\vec{t}$ on A determines a symmetric bilinear form $<,>$ on A given by

$$< a, b >= \vec{t}(ab). \tag{2.2}$$

Suppose A is finite dimensional and let $B = \{b_1, b_2, \ldots, b_s\}$ be a basis of A. A basis $B^* = \{b_1^*, b_2^*, \ldots, b_s^*\}$ of A is dual to B with respect to the form $<,>$ if

$$< b_i^*, b_j >= \delta_{ij}.$$

The *Gram matrix* of A is the matrix

$$G = || < b_i, b_j > ||. \tag{2.3}$$

Suppose that B^* exists and that $C = ||c_{ij}||$ is an $s \times s$ matrix such that

$$b_i^* = \sum_k c_{ik} b_k. \tag{2.4}$$

Then

$$< b_i^*, b_j >= \sum_k c_{ik} < b_k, b_j >= \delta_{ij}.$$

In matrix notation this says that $CG = I_s$. So C must be G^{-1}. Conversely, if $C = G^{-1}$ then defining b_i^* by (2.4) determines a dual basis B^*. This shows that B^* exists if and only if G is invertible and that if it exists it is unique.

(2.5) Proposition. If $\vec{t}$ is a trace on a finite dimensional algebra A with basis $B = \{b_1, b_2, \ldots, b_s\}$ and $<,>$ is given by (2.2) then the Gram matrix G is invertible if and only if $\vec{t}$ is nondegenerate.

Proof. The trace $\vec{t}$ is degenerate if and only if there exists a $b \in A$ such that $\vec{t}(ab) = 0$ for all $a \in A$. This is the same as saying that $\vec{t}(b_i b) = 0$ for each basis element b_i. If $b = \sum_j c_j b_j$, $c_j \in \mathbb{C}$, we have that the c_j satisfy the system of equations

$$\vec{t}(b_i b) = \sum \vec{t}(b_i b_j) c_j = 0.$$

This system has a nontrivial solution if and only if the matrix $G = ||\vec{t}(b_i b_j)||$ is singular. □

Symmetrization

Let A be a finite dimensional algebra with a nondegenerate trace $\vec{t}$ and let B be a basis of A. Let B^* be the dual basis to B with respect to the form $<,>$ given by (2.2). For $g \in B$ let g^* denote the element of B^* such that $\vec{t}(gg^*) = 1$. Let V_1 and V_2 be representations of A of dimensions d_1 and d_2 respectively.

(2.6) Proposition. Let C be any $d_1 \times d_2$ matrix with entries in $\mathbb{C}$. If

$$[C] = \sum_{g \in B} V_1(g) C V_2(g^*),$$

then, for any $a \in A$,

$$V_1(a)[C] = [C]V_2(a).$$

Proof. Let $a \in A$. Then

$$\begin{aligned}
V_1(a)[C] &= \sum_g V_1(ag)CV_2(g^*) \\
&= \sum_{g\in B} V_1(\sum_{h\in B} < ag, h^* > h)CV_2(g^*) \\
&= \sum_{g,h\in B} < ag, h^* > V_1(h)CV_2(g^*) \\
&= \sum_{g,h\in B} V_1(h)C\vec{t}(agh^*)V_2(g^*) \\
&= \sum_{h\in B} V_1(h)C\sum_{g\in B} \vec{t}(h^*ag)V_2(g^*) \\
&= \sum_{h\in B} V_1(h)CV_2(\sum_{g\in B} < h^*a, g > g^*) \\
&= \sum_{h\in B} V_1(h)CV_2(h^*a) \\
&= [C]V_2(a). \quad \square
\end{aligned}$$

If V_1 and V_2 are irreducible then Schur's lemma gives that $[C] = 0$ if V_1 and V_2 are inequivalent and that if $V_1 = V_2$ then $[C] = cId_{d_1}$ for some $c \in \mathbb{C}$.

Let A be a finite dimensional algebra. The action of A on itself by multiplication on the left turns A into an A-module. The resulting representation is the *regular representation* of A and we denote it by $\vec{A}$. The set $\vec{A}$ is the same as the set A, but we distiniguish elements of $\vec{A}$ by writing $\vec{a} \in \vec{A}$. As usual we denote the algebra of this representation by $\vec{A}(A)$. We denote the trace of this representation by tr. Notice that the trace tr of the regular representation can be given by

$$tr(a) = \sum_{g\in B} ag\mid_g, \tag{2.7}$$

where $a \in A$ and B is any basis of A. Here $a\mid_g$ denotes the coefficient of g in the expansion of $a \in A$ in terms of the basis B.

(2.8) Theorem. If A is a finite dimensional algebra such that the regular representation $\vec{A}$ has nondegenerate trace then every representation V of A is completely decomposable.

Proof. Let tr denote the trace of the regular representation. Let B be a basis of A and for each $g \in B$ let g^* denote the element of the dual basis to B with respect to the trace tr such that $tr(gg^*) = 1$.

Let V be a representation of A of dimension d and let V_1 be an irreducible invariant subspace of V. Let $P: V \to V$ be an arbitrary projection of V onto V_1. Define

$$P_1 = \sum_{g\in B} V(g)PV(g^*).$$

Then, by (2.6), we know that

$$V(a)P_1 = P_1V(a).$$

Since V_1 is an A-invariant subspace, $P_1V \subseteq V_1$. Since V_1 is irreducible P_1V is either 0 or V_1.

Let $e = \sum_{g \in B} gg^*$. If $a \in A$ then

$$\begin{aligned} tr(ae) &= tr(\sum_{g \in B} agg^*) \\ &= \sum_{g \in B} < ag, g^* > \\ &= \sum_{g \in B} ag \mid_g \\ &= tr(a). \end{aligned}$$

This shows that $tr(a(e-1)) = 0$ for all $a \in A$. Since tr is nondegenerate we have that

$$e = \sum_{g \in B} gg^* = 1. \tag{2.9}$$

Now let $v \in V_1$. Then since $V(g^*)v \in V_1$ we have

$$\begin{aligned} P_1 v &= \sum_{g \in B} V(g) P(V(g^*)v) \\ &= \sum_{g \in B} V(g) V(g^*) v \\ &= V(\sum_{g \in B} gg^*) v \\ &= V(1) v \\ &= v. \end{aligned}$$

So $P_1 V = V_1$ and $P_1 P_1 V = P_1 V$.

Let $P_1' = I_d - P_1$ and let $V_2 = P_1' V$. Notice that $V(a) P_1' = P_1' V(a)$ for all $a \in A$. So V_2 is an A-invariant subspace of V. Since, for every $v \in V$, $v = P_1 v + (I_d - P_1')v = P_1 v + P_1' v$, we have $V = P_1 V + P_1' V$. If $v \in P_1 V \cap P_1' V$ then $v = P_1 v = P_1 P_1' v = P_1(I_d - P_1')v = 0$. So $P_1 V \cap P_1' V = 0$. Thus we see that $V = P_1 V \oplus P_1' V$.

If $P_1' V$ is irreducible then we are done. If not apply the same process again with $P_1' V$ in place of V. Since V is finite dimensional continuing this process will eventually produce a decomposition of V into irreducible representations. □

Now let A be a finite dimensional algebra such that the trace tr of the regular representation $\overrightarrow{A}$ of A is nondegenerate. Let B be a basis of A and for each $g \in B$ let g^* denote the element of the dual basis to B with respect to the trace tr such that $tr(gg^*) = 1$. Let V be a faithful representation of A. By (2.8) we know that V can be completely decomposed into irreducible representations. Choose a maximal set $\{W_\lambda\}$ of nonisomorphic irreducible representations appearing in the decomposition of V. Let $d_\lambda = \dim W_\lambda$ and define $M_{\vec{d}}(\mathbb{C}) = \oplus_\lambda M_{d_\lambda}(\mathbb{C})$. We view $M_{\vec{d}}(\mathbb{C})$ as an algebra of block diagonal matrices with one $d_\lambda \times d_\lambda$ block for each λ. $V(A) \cong \oplus_\lambda W_\lambda(A)$ is a subalgebra of $M_{\vec{d}}(\mathbb{C})$ in a natural way. Let E_{ij}^λ denote the $d \times d$ matrix with 1 in the (i,j) entry of the λth block and 0 everywhere else and let I_λ be the matrix which is the identity on the λth block and 0 everywhere else.

For each $g \in B$ let $W_{ij}^\lambda(g^*)$ denote the (i,j) entry of the matrix $W_\lambda(g^*)$. Then

$$k\text{th row of } (W_{ji}^\lambda(g^*) W_\lambda(g)) = j\text{th row of } (W_\lambda(g^*) E_{ik}^\lambda W_\lambda(g)).$$

So

$$\begin{aligned} k\text{th row of } (\sum_{g \in B} W_{ji}^\lambda(g^*) W_\lambda(g)) &= j\text{th row of } (\sum_g W_\lambda(g^*) E_{ik}^\lambda W_\lambda(g)) \\ &= j\text{th row of } (cI_\lambda \delta_{ik}). \end{aligned} \tag{2.10}$$

So the ith row of $\sum_g W_{ji}^\lambda(g^*)W_\lambda(g)$ is all zeros except for c in the jth spot and all other rows of $\sum_{g\in B} W_{ji}^\lambda(g^*)W_\lambda(g)$ are zero. So

$$\sum_g W_{ji}^\lambda(g^*)W_\lambda(g) = cE_{ij}^\lambda \tag{2.11}$$

for some $c \in \mathbb{C}$. We can determine c by setting $i = k$ to get

$$\begin{aligned} cd_\lambda &= tr(cI_\lambda \delta_{ii}) \\ &= tr(\sum_g W_\lambda(g^*)E_{ii}^\lambda W_\lambda(g)) \\ &= tr(\sum_g W_\lambda(g)W_\lambda(g^*)E_{ii}) \\ &= tr(W_\lambda(\sum_g gg^*)E_{ii}^\lambda). \end{aligned}$$

Since the trace of the regular representation was used to construct the g^* we have, (2.9), that $\sum_g gg^* = 1$, giving

$$\begin{aligned} tr(W_\lambda(\sum_g gg^*)E_{ii}^\lambda) &= tr(W_\lambda(1)E_{ii}^\lambda) \\ &= tr(I_\lambda E_{ii}^\lambda) \\ &= 1. \end{aligned}$$

So $cd_\lambda = 1$ and we can write (2.11) as

$$d_\lambda \sum_g W_{ji}^\lambda(g^*)W_\lambda(g) = E_{ij}^\lambda.$$

Since we have expressed each E_{ij}^λ as a linear combination of basis elements of $V(A)$ we have that $E_{ij}^\lambda \in V(A)$ for every i and j. But the E_{ij}^λ form a basis of $M_{\vec{d}}(\mathbb{C})$. So $M_{\vec{d}}(\mathbb{C}) \subseteq V(A)$. Then $A \cong V(A) = M_{\vec{d}}(\mathbb{C})$. We have proved the following theorem.

(2.12) Theorem. (Artin-Wedderburn) If A is a finite dimensional algebra such that the trace of the regular representation of A is nondegenerate, then, for some set of positive integers d_λ,

$$A \cong \oplus_\lambda M_{d_\lambda}(\mathbb{C}).$$

Examples

1. Let $\mathcal{A} = \{a_i\}$ and $\mathcal{B} = \{b_i\}$ be two bases of A and let $\mathcal{A}^* = \{a_i^*\}$ and $\mathcal{B}^* = \{b_i^*\}$ be the associated dual bases with respect to a nondegenerate trace $\vec{t}$ on A. Then

$$\begin{aligned} b_i &= \sum_j s_{ij}a_j, \quad \text{and} \\ b_i^* &= \sum_j t_{ij}a_j^*, \end{aligned}$$

for some constants s_{ij} and t_{ij}. Then

$$\begin{aligned} \delta_{ij} =< b_i, b_j^* > &=< \sum_k s_{ik}a_k, \sum_l t_{jl}a_l^* > \\ &= \sum_{k,l} s_{ik}t_{jl} < a_k, a_l^* > \\ &= \sum_{k,l} s_{ik}t_{jl}\delta_{kl} \\ &= \sum_k s_{ik}t_{jk}. \end{aligned}$$

In matrix notation this says that the matrices $S = ||s_{ij}||$ and $T = ||t_{ij}||$ are such that

$$ST^t = I.$$

Then, in the setting of Proposition (2.6),

$$\begin{aligned}\sum_i V_1(b_i)CV_2(b_i^*) &= \sum_i(\sum_j s_{ij}V_1(a_j))C(\sum_k t_{ik}V_2(a_k^*)) \\ &= \sum_{j,k}(\sum_i s_{ij}t_{ik})V_1(a_j)CV_2(a_k^*) \\ &= \sum_{j,k}\delta_{jk}V_1(a_j)CV_2(a_k^*) \\ &= \sum_j V_1(a_j)CV_2(a_j^*).\end{aligned}$$

This shows that the matrix $[C]$ of Proposition (2.6) is independent of the choice of basis.

2. Let A be the algebra of elements of the form $c_1 + c_2e$, $c_1, c_2 \in \mathbb{C}$, where $e^2 = 0$. A is commutative and $\vec{t}$ defined by $\vec{t}(c_1+c_2e) = c_1+c_2$ is a nondegenerate trace on A. The regular representation $\overrightarrow{A}$ of A is not completely decomposable. The subspace $\mathbb{C}\ \overrightarrow{e} \subseteq \overrightarrow{A}$ is invariant and its complementary subspace $\mathbb{C}$ is not. The trace of the regular representation is given explicitly by $tr(1) = 2$ and $tr(e) = 0$. tr is degenerate. There is no matrix representation of A that has trace given by $\vec{t}$.

3. Suppose that G is a finite group and that $A = \mathbb{C}G$ is its group algebra. Then the group elements $g \in G$ form a basis of A. So, using (2.7), the trace of the regular representation can be expressed in the form

$$\begin{aligned}tr(a) &= \sum_{g\in G} ag\ |_g \\ &= \sum_{g\in G} a\ |_1 \\ &= |G|a\ |_1,\end{aligned}$$

where 1 denotes the identity in G and $a\ |_g$ denotes the coefficient of g in a. Since $tr(g^{-1}g) = |G| \neq 0$ for each $g \in G$, tr is nondegenerate. If we set $\vec{t}(a) = a\ |_1$ then $\vec{t}$ is a trace on A and $\{g^{-1}\}_{g\in G}$ is the dual basis to the basis $\{g\}_{g\in G}$ with respect to this trace.

4. Let $\vec{t}$ be the trace of a faithful realization ϕ of an algebra A (i.e. for each $a \in A$, $\vec{t}(a)$ is given by the standard trace of $\phi(a)$ where ϕ is an injective homomorphism $\phi: A \to M_d(\mathbb{C})$). Let $\sqrt{A} = \{a \in A | \vec{t}(ab) = 0 \text{ for all } b \in A\}$. $\sqrt{A}$ is an ideal of A.

Let $a \in \sqrt{A}$. Then $tr(a^{k-1}a) = tr(a^k) = 0$ for all k. If $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $\phi(a)$ then $\vec{t}(a^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_d^k = p_k(\lambda) = 0$ for all $k > 0$, where p_k represents the kth power symmetric function [Mac]. Since the power symmetric functions generate the ring of symmetric functions this means that the elementary symmetric functions $e_k(\lambda) = 0$ for $k > 0$, [Mac] p.17, (2.14′). Since the characteristic polynomial of $\phi(a)$ can be written in the form

$$char_{\phi(a)}(t) = t^d - e_1(\lambda)t^{d-1} + e_2(\lambda)t^{d-2} - \cdots \pm e_d(\lambda),$$

we get that $char_{\phi(a)}(t) = t^d$. But then the Cayley-Hamilton theorem implies that $\phi(a)^d = 0$. Since ϕ is injective we have that $a^d = 0$. So a is nilpotent.

Let J be an ideal of nilpotent elements and suppose that $a \in J$. For every element $b \in A$, $ba \in J$ and ba is nilpotent. This implies that $\phi(ba)$ is nilpotent. By noting that a matrix is nilpotent only if in Jordan block form the diagonal contains all zeros we see that $\vec{t}(ba) = 0$. Thus $a \in \sqrt{A}$.

So $\sqrt{A}$ can be defined as the largest ideal of nilpotent elements. Furthermore, since the regular representation of A is always faithful $\sqrt{A}$ is equal to the set $\{a \in A | tr(ab) = 0 \text{ for all } b \in A\}$ where tr is the trace of the regular representation of A.

5. Let $\mathcal{A}$ be a basis and $\vec{t}$ the trace of a faithful realization of an algebra A as in Ex. 3, and let $G(\mathcal{A})$ be the Gram matrix with respect to the basis $\mathcal{A}$ and the trace $\vec{t}$ as given by (2.2) and (2.3). If $\mathcal{B}$ is another basis of A then

$$G(\mathcal{B}) = P^t G(\mathcal{A}) P,$$

where P is the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$. So the rank of the Gram matrix is independent of the choice of the basis $\mathcal{A}$.

Choose a basis $\{a_1, a_2, \ldots, a_k\}$ of $\sqrt{A}$ ($\sqrt{A}$ defined in Ex. 3) and extend this basis to a basis $\{a_1, a_2, \ldots, a_k, b_1, \ldots, b_s\}$ of A. The Gram matrix with respect to this basis is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & G(B) \end{pmatrix}$$

where $G(B)$ denotes the Gram matrix on $\{b_1, b_2, \ldots, b_s\}$. So the rank of the Gram matrix is certainly less than or equal to s.

Suppose that the rows of $G(B)$ are linearly dependent. Then for some constants $c_1, c_2, \ldots, c_s$, not all zero

$$c_1\vec{t}(b_1 b_i) + c_2\vec{t}(b_2 b_i) + \cdots + c_s\vec{t}(b_s b_i) = 0,$$

for all $1 \leq i \leq s$. So

$$\vec{t}((\sum_j c_j b_j) b_i) = 0, \quad \text{for all } i.$$

This implies that $\sum_j c_j b_j \in \sqrt{A}$. This is a contradiction to the construction of the b_j. So the rows of $G(B)$ are linearly independent.

Thus the rank of the Gram matrix is s or equivalently the corank of the Gram matrix of A is equal to the dimension of the radical $\sqrt{A}$. Thus, the trace tr of the regular representation of A is nondegenerate if and only if $\sqrt{A} = (0)$.

6. Let W be an irreducible representation of an arbitrary algebra A and let $d = \dim W$. Denote $W(A)$ by A_W. Note that representation W is also an irreducible representation of A_W ($W(a) = a$ for all $a \in A_W$).

We show that tr is nondegenerate on A_W, i.e. that if $a \in A_W$, $a \neq 0$, then there exists $b \in A_W$ such that $tr(ba) \neq 0$. Since a is a nonzero matrix there exists some $w \in W$ such that $aw \neq 0$. Now $Aaw \subseteq W$ is an A-invariant subspace of W and not 0 since $aw \neq 0$. Thus $Aaw = W$. So there exists some $b \in A_W$ such that $baw = w$. This shows that ba is not nilpotent. So $tr(ba) \neq 0$. So tr is nondegenerate on A_W. This means that $A_W = \oplus_\lambda M_{d_\lambda}(\mathbb{C})$ for some d_λ. But since by Schur's lemma $A_W = I_d(\mathbb{C})$, where $d = \dim W$, we see that $W(A) = A_W = M_d(\mathbb{C})$.

7. Let A be a finite dimensional algebra and let $\vec{A}$ denote the regular representation of A. The set $\vec{A}$ is the same as the set A, but we distiguish elements of $\vec{A}$ by writing $\vec{a} \in A$.

A linear transformation B of $\vec{A}$ is in the centralizer (as defined by (1.10)) of $\vec{A}$ if for every element $a \in A$ and $\vec{x} \in \vec{A}$,

$$Ba\vec{x} = aB\vec{x}.$$

Let $B\vec{1} = \vec{b}$. Then

$$\begin{aligned} B\vec{a} &= Ba\vec{1} \\ &= aB\vec{1} \\ &= a\vec{b} \\ &= \vec{ab}. \end{aligned}$$

So B acts on $\vec{a} \in \vec{A}$ by right multiplication by b. Conversely, it is easy to see that the action of right multiplication commutes with the action of left multiplication since

$$(a\vec{x})b = a(\vec{x}b),$$

for all $a, b \in A$, and $\vec{x} \in \vec{A}$. So the centralizer algebra of the regular representation is the algebra of matrices determined by the action of right multiplication of elements of A.

Notes and References

The approach to the theory of semisimple algebras that is presented in this section and the following section follows closely a classical approach to the representation theory of finite groups, see for example [Se] or [Ha]. Once one has the analogue of the symmetrization process for finite groups, the only nontrivial step in the theory that is not exactly analogous to the theory for finite groups is formula (2.9).

I discovered this method after reading the sections of [CR1] concerning Frobenius and symmetric algebras. Frobenius and symmetric algebras were introduced by R. Brauer and C. Nesbitt, [BN] and [Ns]. T. Nakayama [Nk] has a version of Theorem (2.6) and R. Brauer [Br] proves analogues of the Schur orthogonality relations that are analogous to formula (2.10). Ikeda [Ik], and Higman [Hg], following work of Gaschütz [Ga], construct "Casimir" type elements similar to those in (2.9) and §3 Ex. 7. In [CR2] §9 Curtis and Reiner use a similar approach but with different proofs, communicated to them by R. Kilmoyer, to obtain theorems (3.8) and (3.9) for split semisimple algebras (over fields of characteristic 0). N. Wallach has told me that essentially the same approach works for finite dimensional Lie algebras.

This approach is useful for studying semisimple algebras that have distiguished bases. The recent interest in quantum deformations is producing a host of examples of semisimple algebras that are *not* group algebras but that *do* have distinguished bases. Some examples are Hecke algebras associated to root systems, the Brauer algebra, and the Birman-Wenzl algebra [BW]. For an approach to the Hecke algebras that is essentially an application of the general theory given here see [GU] and [Cr3] §68C.

I would like to thank Prof. A. Garsia for suggesting that I try to find an analogue of the symmetrization process for finite groups for the Brauer algebra. It was this problem that resulted in my discovery of this approach. I would like to thank Prof. C.W. Curtis for his helpful suggestions in locating literature with a similar approach. I would also like to thank Prof. Garsia for showing me the proofs of Exs. 4 and 5.

3. Semisimple algebras

An algebra A is *simple* if $A \cong M_d(\mathbb{C})$.
Suppose $\vec{t}$ is a trace on $M_d(\mathbb{C})$. Then

$$\begin{aligned}\vec{t}(E_{ij}) &= \vec{t}(E_{i1}E_{1j}) \\ &= \vec{t}(E_{1j}E_{i1}) \\ &= \vec{t}(E_{11})\delta_{ij}.\end{aligned}$$

If $a = \|a_{ij}\| \in M_d(\mathbb{C})$ then

$$\begin{aligned}\vec{t}(a) &= \vec{t}(\sum_{i,j} a_{ij}E_{ij}) \\ &= \sum_{i,j} a_{ij}\vec{t}(E_{ij}) \\ &= \sum_{i,j} a_{ij}\vec{t}(E_{11})\delta_{ij} \\ &= \vec{t}(E_{11})(\sum_i a_{ii}).\end{aligned}$$

So, up to a constant factor there is a unique trace function on $M_d(\mathbb{C})$, that given by the standard trace on matrices.

Suppose J is an ideal of $M_d(\mathbb{C})$ and that $a = \|a_{ij}\| \in J$, with $a \neq 0$. So $a_{ij} \neq 0$ for some (i,j). Since $a \in J$ and J is an ideal,

$$(1/a_{ij})\sum_{k=1}^{d} E_{ki}aE_{jk} = (1/a_{ij})\sum_k a_{ij}E_{kk} = I_d$$

is an element of J. Thus $J = M_d(\mathbb{C})$. This shows that the only ideals of $M_d(\mathbb{C})$ are the trivial ones, 0 and $M_d(\mathbb{C})$. It is an immediate consequence of §1 Ex. 1 that the center of $M_d(\mathbb{C})$ is $I_d(\mathbb{C}) = \mathbb{C}\, I_d$. Furthermore, I_d is the unique central idempotent in $M_d(\mathbb{C})$.

(3.1) Proposition. There is a unique irreducible representation of $M_d(\mathbb{C})$ given by the usual multiplication of $d \times d$ matrices on all column vectors of size d.

Proof. Let V be the d dimensional vector space of column vectors of size d. The standard basis of V consists of the vectors $e_i = (0,\ldots,0,1,0,\ldots,0)^t, 1 \le i \le d$ where the 1 appears in the ith spot. Suppose that V' is a nonzero invariant subspace of V. Let $v = \sum_i v_ie_i$, $v_i \in \mathbb{C}$, be a nonzero element of V'. So $v_i \neq 0$ for some $1 \le i \le d$. Then $(1/v_i)E_{ji}v = e_j$. Since V' is invariant we have that $e_j \in V'$ for each $1 \le j \le d$. But since the e_j are a basis of V this implies that $V = V'$. So V is an irreducible representation of $M_d(\mathbb{C})$.

Now let W be an arbitrary irreducible representation of $A = M_d(\mathbb{C})$. There is some vector $w \in W$ and some $a \in A$ such that $aw \neq 0$, otherwise W would be the zero representation. If $a = \|a_{ij}\|$ then $aw = \sum_{i,j} a_{ij}E_{ij}w \neq 0$ implies that $E_{ij}w \neq 0$ for some pair (i,j). The space $M_d(\mathbb{C})E_{ij}$ consists of all matrices that are 0 except in the jth column and is isomorphic to V. The map

$$\begin{array}{rccc}\phi: & M_d(\mathbb{C})E_{ij} & \longrightarrow & W \\ & aE_{ij} & \longmapsto & aE_{ij}\end{array} \tag{3.2}$$

is an isomorphism since both V and W are irreducible. □

So the regular representation of $M_d(\mathbb{C})$ decomposes as a direct sum of d copies of the unique irreducible representation V of $M_d(\mathbb{C})$, one copy for each column in $M_d(\mathbb{C})$.

An algebra A is *semisimple* if

$$A \cong \oplus_{\lambda\in\Lambda} M_{d_\lambda}(\mathbb{C}), \tag{3.3}$$

where Λ is a finite index set. The vector $\vec{d} = (d_\lambda), \lambda \in \Lambda$ of positive integers is called the dimension vector of the algebra A. We will use $M_{\vec{d}}(\mathbb{C})$ as a shorthand notation for the algebra given by the right hand side of (3.3). We can view $M_{\vec{d}}(\mathbb{C})$ as the full algebra of block diagonal matrices where the λth block is dimension

d_λ. We denote the matrix having 1 in the (i,j)th position of the λth block and zeros everywhere else by E^λ_{ij}. Denote the matrix which is the identity on the λth block and 0 everywhere else by I_λ.

Any trace on $\oplus_{\lambda\in\Lambda}M_{d_\lambda}(\mathbb{C})$ is completely determined by a vector $\vec{t}=(t_\lambda)$ of complex numbers such that $\vec{t}(E^\lambda_{11})=t_\lambda$ for each λ in the finite index set Λ. The vector $\vec{t}=(t_\lambda)$ is the *trace vector* of the trace $\vec{t}$. A trace $\vec{t}$ on $M_{\vec{d}}(\mathbb{C})$ is nondegenerate if and only if $t_\lambda \neq 0$ for all $\lambda\in\Lambda$. The only ideals of $M_{\vec{d}}(\mathbb{C})=\oplus_{\lambda\in\Lambda}M_{d_\lambda}(\mathbb{C})$ are of the form $\oplus_{\lambda\in\Lambda'}M_{d_\lambda}(\mathbb{C})$ where $\Lambda'\subseteq\Lambda$. The I_λ, $\lambda\in\Lambda$ form a basis of the center of $M_{\vec{d}}(\mathbb{C})$. Every central idempotent is a sum of some subset of the I_λ. There is, up to isomorphism, one irreducible representation of $\oplus_{\lambda\in\Lambda}M_{d_\lambda}(\mathbb{C})$ for each $\lambda\in\Lambda$. It can be given by left multiplication on the space $M_{\vec{d}}(\mathbb{C})E^\lambda_{ij}$, for any i,j, $1\le i,j\le d_\lambda$. The decomposition of the regular representation of $\oplus_{\lambda\in\Lambda}M_{d_\lambda}(\mathbb{C})$ into irreducibles is given by

$$\overrightarrow{M_{\vec{d}}(\mathbb{C})}=\oplus_{\lambda\in\Lambda}W_\lambda^{\oplus d_\lambda} \tag{3.4),}$$

where W_λ denotes the irreducible representation corresponding to λ.

Matrix units and characters

Let A be an algebra and $\hat{A}$ a finite index set such that $A\cong\oplus_{\lambda\in\hat{A}}M_{d_\lambda}(\mathbb{C})$ under an isomorphism $\phi: A\to\oplus_{\lambda\in\hat{A}}M_{d_\lambda}(\mathbb{C})$. (Let $M_{\vec{d}}(\mathbb{C})$ denote the algebra $\oplus_\lambda M_{d_\lambda}(\mathbb{C})$.) Warning: The isomorphism ϕ is not unique; nontrivial automorphisms of $M_{\vec{d}}(\mathbb{C})$ do exist, just conjugate by an invertible matrix. $z_\lambda=\phi^{-1}(I_\lambda)$ is an idempotent and an element of the center of A. The z_λ are the *minimal central idempotents* of A. They are minimal in the sense that every central idempotent of A is a sum of z_λ's. These elements are independent of the isomorphism ϕ.

A set of elements $e^\lambda_{ij}\in A, \lambda\in\hat{A}, 1\le i,j\le d_\lambda$ is a set of *matrix units* of A if

$$e^\lambda_{ij}e^\mu_{rs}=\begin{cases}0, & \text{if } \lambda\neq\mu;\\ 0, & \text{if } \lambda=\mu, j\neq r;\\ e^\lambda_{is}, & \text{if } \lambda=\mu, j=r.\end{cases} \tag{3.5}$$

A *complete set of matrix units* of A is a set of matrix units which forms a basis of A. Let $E^\lambda_{ij}\in M_{\vec{d}}(\mathbb{C})$ denote the matrix having 1 in the (i,j)th position of the λth block and zeros everywhere else. If $\{e^\lambda_{ij}\}$ is a set of matrix units of A, the mapping $e^\lambda_{ij}\mapsto E^\lambda_{ij}$ determines explicitly an isomorphism $A\to M_{\vec{d}}(\mathbb{C})$. Conversely, an ismorphism $\phi: A\to M_{\vec{d}}(\mathbb{C})$ determines a set of matrix units $e^\lambda_{ij}=\phi^{-1}(E^\lambda_{ij})$. Note that the e^λ_{ii} are minimal orthogonal idempotents in A.

Let W_λ, $\lambda\in\hat{A}$ denote the irreducible representations A. By (3.1) and (3.2), for each $\lambda\in\hat{A}$,

$$W_\lambda\cong Ae^\lambda_{ij}, \tag{3.6}$$

for any i,j, $1\le i,j\le d_\lambda$, where the action of A on Ae^λ_{ij} is given by left multiplication. For each $\lambda\in\hat{A}$ denote the character of the irreducible representation W_λ by χ^λ and for each $\lambda\in\hat{A}$ and $a\in A$ let $W^\lambda_{ij}(a)$ denote the (i,j)th entry of the matrix $W_\lambda(a)$. Note that we can view each $W_\lambda(a)$ as a matrix in $M_{\vec{d}}(\mathbb{C})$ with all but the λth block 0.

Let B be an arbitrary basis of A. Let $\vec{t}=(t_\lambda), \lambda\in\hat{A}$ be a nondegenerate trace on A. For each $g\in B$ let g^* denote the element of the dual basis to B with respect to the trace $\vec{t}$ such that $\vec{t}(gg^*)=1$.

(3.7) **Theorem.** (Fourier inversion formula) The elements

$$e^\lambda_{ij}=\sum_{g\in B}t_\lambda W^\lambda_{ji}(g^*)g$$

form a complete set of matrix units of A.

Proof. Let $\phi: A\to M_{\vec{d}}(\mathbb{C})$ be given by $\phi(a)=\oplus_\lambda W_\lambda(a)$. This is an isomorphism. For each $\lambda\in\hat{A}$ and $1\le i,j\le d_\lambda$ let

$$e^\lambda_{ij}=\phi^{-1}(E^\lambda_{ij}).$$

The set $B = \{e^\lambda_{ij}\}$ forms a basis of A. The dual basis with respect to the trace $\vec{t} = (t_\lambda)$ is the basis $\{(1/t_\lambda)e^\lambda_{ji}\}$.

$$\begin{aligned}\sum_{g\in B} t_\lambda W^\lambda_{ji}(g^*)g &= \sum_{k,l,\mu} t_\lambda W^\lambda_{ji}((1/t_\mu)e^\mu_{lk})e^\mu_{kl}\\ &= \sum_{k,l,\mu} t_\lambda(1/t_\mu)\delta_{jl}\delta_{ik}\delta_{\lambda\mu}e^\mu_{kl}\\ &= e^\lambda_{ij}.\end{aligned}$$

Notice that

$$k\text{th row of } (\sum_{g\in B} W^\lambda_{ji}(g^*)W_\lambda(g)) = j\text{th row of } (\sum_{g\in B} W_\lambda(g^*)E^\lambda_{ik}W_\lambda(g)).$$

By §2 Ex. 1 we know that $\sum_{g\in B} W_\lambda(g^*)E^\lambda_{ik}W_\lambda(g)$ is independent of the basis B. □

(3.8) Theorem.

$$\sum_{g\in B} t_\lambda\chi^\lambda(g^*)g = z_\lambda.$$

Proof.

$$\begin{aligned}z_\lambda &= \sum_{i=1}^{d_\lambda} e^\lambda_{ii}\\ &= \sum_i\sum_{g\in B} t_\lambda W^\lambda_{ii}(g^*)g\\ &= \sum_{g\in B}(\sum_i t_\lambda W^\lambda_{ii}(g^*))g\\ &= \sum_{g\in B} t_\lambda\chi^\lambda(g^*)g. \qquad \square\end{aligned}$$

(3.9) Theorem.

$$\sum_{g\in B}\chi^\lambda(g)\chi^\mu(g^*) = (d_\lambda/t_\lambda)\delta_{\lambda\mu}.$$

Proof.

$$\begin{aligned}d_\lambda\delta_{\lambda\mu} &= \chi^\lambda(z_\mu)\\ &= \chi^\lambda(\sum_{g\in B} t_\lambda\chi^\mu(g^*)g)\\ &= \sum_{g\in B} t_\lambda\chi^\lambda(g)\chi^\mu(g^*). \qquad \square\end{aligned}$$

Examples.

1. If A is commutative and semisimple then all irreducible representations of A are one dimensional. This is not necessarily true for algebras over fields which are not algebraically closed (since Schur's lemma takes a different form).

2. If R is a ring with identity and $M_n(R)$ denotes $n\times n$ matrices with entries in R, the ideals of $M_n(R)$ are of the form $M_n(I)$ where I is an ideal of R.

3. If V is a vector space over $\mathbb{C}$ and V^* is the space of $\mathbb{C}$ valued functions on V then $\dim V^* = \dim V$. If B is a basis of V then the functions δ_b, $b\in B$, determined by

$$\delta_b(b_i) = \begin{cases} 1, & \text{if } b = b_i;\\ 0, & \text{otherwise;}\end{cases}$$

for $b_i \in B$, form a basis of V^*. If A is a semisimple algebra isomorphic to $M_{\vec{d}}(\mathbb{C}) = \oplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$, $\hat{A}$ an index set for the irreducible representations W_λ of A, then

$$\dim A = \sum_{\lambda \in \hat{A}} d_\lambda^2, \tag{3.10}$$

and the functions W_{ij}^λ ($W_{ij}^\lambda(a)$ the (i,j)th entry of the matrix $W_\lambda(a)$, $a \in A$) on A form a basis of A^*. The W_{ij}^λ are simply the functions $\delta_{e_{ij}^\lambda}$ for an appropriate set of matrix units $\{e_{ij}^\lambda\}$ of A. This shows that the coordinate functions of the irreducible representations are linearly independent. Since $\chi^\lambda = \sum_i W_{ii}^\lambda$, the irreducible characters are are also linearly independent.

4. Let A be a semisimple algebra. Virtual characters are elements of the vector space $R(A)$ consisting of the $\mathbb{C}$-linear span of the irreducible characters of A. We know that there is a one-to-one correspondence between the minimal central idempotents of A and the irreducible characters of A. Since the minimal central idempotents of A form a basis of the center $Z(A)$ of A, we can define a vector space isomorphism $\phi: Z(A) \to R(A)$ by setting $\phi(z_\lambda) = \chi^\lambda$ for each $\lambda \in \hat{A}$ and extending linearly to all of $Z(A)$.

Given a trace $\vec{t}$ be a nondegenerate trace on A with trace vector (t_λ) it is more natural to define ϕ by setting $\phi(z_\lambda/t_\lambda) = \chi^\lambda$. Then, for $z \in Z(A)$,

$$\phi(z)(a) = \vec{t}(za), \tag{3.11}$$

since

$$\begin{aligned}\phi(z_\mu/t_\mu)(a) &= \vec{t}(z_\mu/t_\mu a)\\ &= \vec{t}((1/t_\mu)z_\mu a)\\ &= (1/t_\mu)(t_\mu\chi^\mu(a))\\ &= \chi^\mu(a).\end{aligned}$$

5. If A is a semisimple algebra isomorphic to $M_{\vec{d}}(\mathbb{C}) = \oplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$, $\hat{A}$ an index set for the irreducible representations W_λ of A, then the regular representation decomposes as

$$\vec{A} \cong \oplus_{\lambda \in \hat{A}} W_\lambda^{\oplus d_\lambda}.$$

If matrix units e_{ij}^λ are given by (3.7) then

$$tr(e_{ii}^\lambda) = tr(d_\lambda E_{ii}^\lambda) = d_\lambda.$$

So the trace of the regular representation of A, tr, is given by the trace vector $\vec{t} = (t_\lambda)$ where $t_\lambda = d_\lambda$ for each $\lambda \in \hat{A}$.

6. Let A be a semisimple algebra and let $B^* = \{g^*\}$ be a dual basis to a basis $B = \{g\}$ of A with respect to the trace of the regular representation of A. We can define an inner product on the space $R(A)$ of virtual characters (Ex. 4) of A by

$$<\chi, \chi'> = \sum_{g \in B} \chi(g)\chi'(g^*).$$

The irreducible characters of A are orthonormal with respect to this inner product. Note that if χ, χ' are the characters of representations V and V' respectively, then, by Ex. 4 and Theorem (3.9),

$$<\chi, \chi'> = \dim \mathrm{Hom}_A(V, V').$$

If χ^λ is the character of the irreducible representation W_λ of A then $<\chi^\lambda, \chi>$ gives the multiplicity of W_λ in the representation V as in §1 Ex. 3.

7. Let A be a semisimple algebra and $\vec{t} = (t_\lambda)$ be a nondegenerate trace on A. Let B be a basis of A and for each $g \in B$ let g^* denote the element of the dual basis to B with respect to the trace $\vec{t}$ such that $\vec{t}(gg^*) = 1$. For each $a \in A$ define

$$[a] = \sum_{g \in B} gag^*.$$

By §2 Ex. 1 the element $[a]$ is independent of the choice of the basis B. By using as basis a set of matrix units e^λ_{ij} of A we get

$$\begin{aligned}
[a] &= \sum_{i,j,\lambda} (1/t_\lambda) e^\lambda_{ij} a e^\lambda_{ji} \\
&= \sum_{i,j,\lambda} (1/t_\lambda) a^\lambda_{jj} e^\lambda_{ii} \\
&= \sum_\lambda (1/t_\lambda) (\sum_j a^\lambda_{jj} (\sum_i e^\lambda_{ii})) \\
&= \sum_\lambda (1/t_\lambda) \chi^\lambda(a) z_\lambda.
\end{aligned} \tag{3.12}$$

So $\chi^\lambda([a]) = (d_\lambda/t_\lambda)\chi^\lambda(a)$. By (3.9)

$$\begin{aligned}
\sum_{g \in B} (t_\lambda^2/d_\lambda)\chi^\mu(g^*)[g] &= \sum_\lambda \sum_{g \in B} (t_\lambda^2/d_\lambda)(1/t_\lambda)\chi^\lambda(g)\chi^\mu(g^*) z_\lambda \\
&= \sum_\lambda \delta_{\lambda\mu} z_\lambda \\
&= z_\mu.
\end{aligned} \tag{3.13}$$

Thus the $[g]$, $g \in B$, span the center of A.

8. Let G be a finite group and let $A = \mathbb{C}G$. Let $\vec{t}$ be the trace on A given by

$$\vec{t}(a) = a|_1,$$

where 1 is the identity in G. By Ex. 5 and §2 Ex. 3 the trace vector of $\vec{t}$ is given by $t_\lambda = (d_\lambda/|G|)$ where d_λ is the dimension of the irreducible representation of G corresponding to λ.

If $h \in G$, then the element

$$[h] = \sum_{g \in B} ghg^* = \sum_{g \in B} ghg^{-1}$$

is a multiple of the sum of the elements of G that are conjugate to h. Let Λ be an index set for the conjugacy classes of G and, for each $\lambda \in \Lambda$, let C_λ denote the sum of the elements in the conjugacy class indexed by λ. The C_λ are linearly independent elements of $\mathbb{C}G$. Furthermore, by Ex. 7 they span the center of $\mathbb{C}G$. Thus Λ must also be an index set for the irreducible representations of G. So we see that the irreducible representations of the group algebra of a finite group are indexed by conjugacy classes.

9. Let G be a finite group. Let C_λ denote the conjugacy classes of G. Note that since

$$tr(V(hgh^{-1})) = tr(V(h)V(g)V(h)^{-1}) = tr(V(g))$$

for any representation V of G and all $g, h \in G$, characters of G are constant on conjugacy classes. Using Theorem (3.8),

$$\begin{aligned}
|G|\delta_{\lambda\mu} &= \sum_g \chi^\lambda(g)\chi^\mu(g^{-1}) \\
&= \sum_\rho \sum_{g \in C_\rho} \chi^\lambda(g)\chi^\mu(g^{-1}) \\
&= \sum_\rho |C_\rho| \chi^\lambda(\rho)\chi^\mu(\rho'),
\end{aligned}$$

where ρ' is such that $C_{\rho'}$ is the conjugacy class which contains the inverses of the elements in C_ρ. Define matrices $\Xi = ||\Xi_{\lambda\rho}||$ and $\Xi' = ||\Xi'_{\lambda\rho}||$ by $\Xi_{\lambda\rho} = \chi^\lambda(\rho)$ and $\Xi'_{\lambda\rho} = |C_\rho|\chi^\lambda(\rho')$. By Ex. 8 these matrices are square. In matrix notation the above is

$$\Xi\Xi'^t = |G|I.$$

But then we also have that $\Xi'^t\Xi = |G|I$, or equivalently that

$$\sum_\lambda \chi^\lambda(\rho')\chi^\lambda(\tau) = (|G|/|C_\rho|)\delta_{\rho\tau}.$$

10. This example gives a generalization of the previous example. Let A be a semisimple algebra and suppose that B is a basis of A and that there is a partition of B into classes such that if b and $b' \in B$ are in the same class then for every $\lambda \in \hat{A}$,

$$\chi^\lambda(b) = \chi^\lambda(b'). \tag{3.14}$$

The fact that the characters are linearly independent implies that the number of classes must be the same as the number of irreducible characters χ^λ. Thus we can index the classes of B by the elements of $\hat{A}$. Assume that we have fixed such a correspondence and denote the classes of B by C_λ, $\lambda \in \hat{A}$.

Let $\vec{t}$ be a nondegenerate trace on A and let G be the Gram matrix (2.3) with respect to the basis B and the trace $\vec{t}$. If $g \in B$, let g^* denote the element of the dual basis to B, with respect to the trace $\vec{t}$, such that $\vec{t}(gg^*) = 1$. Let $G^{-1} = C = ||c_{gg'}||$ and recall (2.4) that $g^* = \sum_{g'\in B} c_{gg'}g'$. Then

$$\begin{aligned}(d_\lambda/t_\lambda)\delta_{\lambda\mu} &= \sum_{g\in B} \chi^\lambda(g)\chi^\mu(g^*)\\ &= \sum_{g\in B}\chi^\lambda(g)\chi^\mu(\sum_{g'\in B} c_{gg'}g')\\ &= \sum_{g,g'\in B}\chi^\lambda(g)c_{gg'}\chi^\mu(g').\end{aligned}$$

Collecting $g, g' \in B$ by class gives

$$\begin{aligned}(d_\lambda/t_\lambda)\delta_{\lambda\mu} &= \sum_{\rho,\tau}\sum_{\substack{g\in C_\rho\\ g'\in C_\tau}} \chi^\lambda(g)c_{gg'}\chi^\mu(g')\\ &= \sum_{\rho,\tau}\sum_{\substack{g\in C_\rho\\ g'\in C_\tau}} \chi^\lambda(\rho)c_{gg'}\chi^\mu(\tau),\end{aligned}$$

where $\chi^\lambda(\rho)$ denotes the value of the character χ^λ at elements of the class C_ρ. Now define a matrix $\overline{C} = ||\bar{c}_{\rho\tau}||$ with entries

$$\bar{c}_{\rho\tau} = \sum_{\substack{g\in C_\rho\\ g'\in C_\tau}} c_{gg'},$$

and let $\Xi = ||\Xi_{\lambda\rho}||$ and $\Xi' = ||\Xi'_{\lambda\rho}||$ be matrices given by $\Xi_{\lambda\rho} = \chi^\lambda(\rho)$ and $\Xi'_{\lambda\rho} = (t_\lambda/d_\lambda)\chi^\lambda(\rho)$. Note that all of these matrices are square. Then the above gives that

$$I = \Xi\overline{C}\Xi'^t.$$

So

$$I = \overline{C}\Xi'^t\Xi,$$

or equivalently that

$$\begin{aligned}
\delta_{\rho\tau} &= \sum_{\sigma,\lambda} \bar{c}_{\rho\sigma}(t_\lambda/d_\lambda)\chi^\lambda(\sigma)\chi^\lambda(\tau) \\
&= \sum_{\sigma,\lambda} \sum_{\substack{g\in C_\rho \\ g'\in C_\sigma}} c_{gg'}(t_\lambda/d_\lambda)\chi^\lambda(\sigma)\chi^\lambda(\tau) \\
&= \sum_{\lambda} \sum_{g\in C_\rho} \sum_{g'\in B} c_{gg'}\chi^\lambda(\sigma)\chi^\lambda(\tau) \\
&= \sum_{g\in C_\rho} (\sum_{\lambda} \chi^\lambda(g^*)\chi^\lambda(\tau)).
\end{aligned}$$

Notes and References

The Fourier Inversion formula for representations of finite groups appears in [Se] p. 49. I must thank Prof. A. Garsia for suggesting the problem of finding a generalization. I know of no references giving a similar generalization. Theorems (3.7) and (3.8) are due to R. Kilmoyer and appear in [CR2] (9.17) and (9.19). Ex. 3 is the Frobenius-Schur theorem. Ex. 9 is known as the second orthogonality relation for characters of finite groups (the first orthogonality relation being (3.8)), see [CR2] (9.26) or [Se] Chap. 2, Prop. 7. The generalization given in Ex. 10 is new as far as I know. [R1] shows that the Brauer algebra is an example of semsimple algebra that is not a group algebra with a natural basis that can be partitioned into classes such that (3.13) holds.

4. Double centralizer nonsense

Tensor products

If P and Q are two matrices with entries from $\mathbb{C}$, then the *tensor product* of P and Q is the matrix

$$P \otimes Q = \|p_{ij} Q\|, \tag{4.1}$$

where p_{ij} denotes the (i,j)th entry in P. If V and W are two vector spaces with bases $B_V = \{v_i\}$ and $B_W = \{w_i\}$ respectively, the *tensor product* $V \otimes W$ is the vector space consisting of the linear span of the words $v_i w_j$. If V is dimension n and W is dimension m, then $V \otimes W$ is dimension nm. In general, for any $v \in V$ and $w \in W$, the word vw can be expressed in terms of the words $v_i w_j$ by using linearity, i.e. for all $c, d \in \mathbb{C}$, $v_i, v_j \in B_V$ and $w_r, w_s \in B_W$,

$$(cv_i + dv_j)w_r = cv_i w_r + dv_j w_r \quad \text{and}$$
$$v_i(cw_r + dw_s) = cv_i w_r + dv_i w_s.$$

Suppose that A and C are two arbitrary algebras. We can define an algebra structure on the vector space $A \otimes C$ (we distinguish the tensor product of algebras from the vector space case by writing (a,c) instead of ac for a word in $A \otimes C$, $a \in A, c \in C$) by defining multiplication of elements of $A \otimes C$ by

$$(a_1, c_1)(a_2, c_2) = (a_1 a_2, c_1 c_2), \tag{4.2}$$

for all $a_1, a_2 \in A$ and $c_1, c_2 \in C$, and extending linearly.

Suppose that V and W are representations of A and C respectively. Define an action of $A \otimes C$ on the vector space $V \otimes W$ by

$$(a,c)(vw) = (av)(cw) \tag{4.3}$$

for all (a,c) and vw, $a \in A, c \in C$, $v \in V$, $w \in W$. This defines a representation of $A \otimes C$ on $V \otimes W$ under which the action of $(a,c), a \in A, c \in C$ on $V \otimes W$ is given by the matrix

$$V(a) \otimes W(c).$$

Centralizer of a completely decomposable representation

Let V be a completely decomposable representation of an algebra A. Assume that

$$V \cong \oplus_{\lambda=1}^{n} W_\lambda^{\oplus m_\lambda},$$

where the W_i are nonisomorphic irreducible representations of V. This means that we can decompose V into irreducible subspaces $V_{\lambda j}$, $1 \le \lambda \le n$, $1 \le j \le m_\lambda$, so that

$$V = \oplus_{\lambda, j} V_{\lambda j},$$

where for each λ and j, $V_{\lambda j} \cong W_\lambda$. Let $d_\lambda = \dim W_i$. Choosing a basis on each of the $V_{\lambda j}$ gives a basis of V which we denote $\mathcal{B}$. Using the basis $\mathcal{B}$ of V, the algebra of the representation V is

$$V(A) = \begin{pmatrix} I_{m_1}(M_{d_1}(\mathbb{C})) & 0 & 0 & \cdots & 0 \\ 0 & I_{m_2}(M_{d_2}(\mathbb{C})) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & I_{m_n}(M_{d_n}(\mathbb{C})) \end{pmatrix}. \tag{4.4}$$

(1.11) shows that the algebra of matrices that commute with all matrices in $V(A)$, is

$$\overline{V(A)} = \begin{pmatrix} M_{m_1}(\overline{W_1(A)}) & 0 & 0 & \cdots & 0 \\ 0 & M_{m_2}(\overline{W_2(A)}) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & M_{m_n}(\overline{W_n(A)}) \end{pmatrix}.$$

Since, by Schur's Lemma, $\overline{W_\lambda(A)} = I_{d_\lambda}(\mathbb{C})$, we get that

$$\overline{V(A)} = \begin{pmatrix} M_{m_1}(I_{d_1}(\mathbb{C})) & 0 & 0 & \cdots & 0 \\ 0 & M_{m_2}(I_{d_2}(\mathbb{C})) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & M_{m_n}(I_{d_n}(\mathbb{C})) \end{pmatrix}. \tag{4.5}$$

(4.6) Theorem. If a representation V of an algebra A is completely decomposable in the form

$$V \cong \oplus_{\lambda=1}^{n} W_\lambda^{\oplus m_\lambda},$$

where the W_λ are nonisomorphic irreducibles, then the centralizer $\overline{V(A)}$ of $V(A)$ is semsimple and

$$\overline{V(A)} \cong \oplus_{\lambda=1}^{n} M_{m_\lambda}(\mathbb{C}).$$

Proof. By a change of basis on V we can put the matrices of (4.5) in the form

$$\begin{pmatrix} I_{d_1}(M_{m_1}(\mathbb{C})) & 0 & 0 & \cdots & 0 \\ 0 & I_{d_2}(M_{m_2}(\mathbb{C})) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & I_{d_n}(M_{m_n}(\mathbb{C})) \end{pmatrix}. \tag{4.7}$$

These matrices are of exactly the same form as those in (4.4) except that the d_λs and m_λs are switched!! (4.7) shows that $\overline{V(A)} \cong \oplus_{\lambda=1}^{n} M_{m_\lambda}(\mathbb{C})$ as algebras. □

Let B be an algebra with an action on V such that $V(B) = \overline{V(A)}$. Let B' be the kernel of the action of B on V and let C be the quotient B/B' so that the induced action of C on V is injective. $C \cong V(B) = \overline{V(A)}$.

From (4.5) we see that with respect to the basis $\mathcal{B}$ on V the action of an element $q \in C$ is given by a matrix of the form

$$\begin{pmatrix} Q_1 \otimes I_{m_1} & 0 & 0 & \cdots & 0 \\ 0 & Q_2 \otimes I_{m_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & Q_n \otimes I_{m_n} \end{pmatrix}, \tag{4.8}$$

where $Q_\lambda \in M_{m_\lambda}(\mathbb{C})$. This action determines a map

$$\begin{array}{cccc} \phi: & C & \longrightarrow & \oplus_\lambda M_{m_\lambda}(\mathbb{C}) \\ & q & \longmapsto & \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & Q_n \end{pmatrix}, \end{array}$$

which, by Theorem (4.6), is an isomorphism. Note that, for each λ, the map

$$\begin{array}{cccc} C_\lambda: & C & \longrightarrow & M_{m_\lambda}(\mathbb{C}) \\ & q & \longmapsto & Q_\lambda \end{array} \tag{4.9}$$

is an irreducible representation of C.

Let E_{ij}^λ denote the matrix in $\oplus_\lambda M_{m_\lambda}(\mathbb{C})$ that is 1 in the (i,j)th entry of the λth block and 0 everywhere else. Define a set of matrix units $e_{ij}^\lambda, 1 \le \lambda \le n, 1 \le i,j \le m_\lambda$ in C by

$$e_{ij}^\lambda = \phi^{-1}(E_{ij}^\lambda).$$

The action of the element e_{ii}^λ on V, is given by the matrix $E_{ii}^\lambda \otimes I_{\bar{m}} \in \overline{V(A)}$. The action of this matrix on V is the projection $p: V \to V_{\lambda i}$;

$$V_{\lambda i} = e_{ii}^\lambda V.$$

Conversely, if $\{e_{ij}^\lambda\}$ is a set of matrix units of C, then, since $1 = \sum_{\lambda,i} e_{ii}^\lambda$ as an element of C, we have a decomposition

$$V = 1 \cdot V = (\sum_{\lambda,i} e_{ii}^\lambda)V = \sum_{\lambda,i} e_{ii}^\lambda V.$$

Since the action of A on V commutes with the action of C we have that $ae_{ii}^\lambda V = e_{ii}^\lambda aV \subset e_{ii}^\lambda V$ for all $a \in A$, showing that each of the spaces $e_{ii}^\lambda V$ is A-invariant. Since, §1 Ex. 5, $e_{ii}^\lambda V \cap e_{jj}^\mu V = \emptyset$ unless $\lambda = \mu$ and $i = j$, the decomposition given above is a direct sum decomposition of V. This decomposition is a decomposition of V into irreducible subspaces under the action of A,

$$V = \oplus_{\lambda,i} e_{ii}^\lambda V. \tag{4.10}$$

Define an action of $C \otimes A$ on V by

$$(q, a)v = qav$$

where $(q, a) \in C \otimes A$ and $v \in V$. Since the actions of C and A on V commute this action is well defined and makes V into an $C \otimes A$ representation. Theorem (4.6) shows that the irreducible representations of C are in one to one correspondence with the irreducible representations of A appearing in the decomposition of V. Let C_λ denote the irreducible representation of C corresponding to λ.

(4.11) Theorem. As $C \otimes A$ representations,

$$V \cong \oplus_{\lambda=1}^n C_\lambda \otimes W_\lambda.$$

Proof. With respect to the basis $\mathcal{B}$ of V the action of $(q, a) \in C \otimes A$ on V is given by the matrix product

$$\begin{pmatrix} Q_1 \otimes I_{m_1} & 0 & \cdots & 0 \\ 0 & Q_2 \otimes I_{m_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & Q_n \otimes I_{m_n} \end{pmatrix} \begin{pmatrix} I_{m_1} \otimes W_1(a) & 0 & \cdots & 0 \\ 0 & I_{m_2} \otimes W_2(a) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & I_{m_n} \otimes W_n(a) \end{pmatrix},$$

which is equal to

$$\begin{pmatrix} Q_1 \otimes W_1(a) & 0 & 0 & \cdots & 0 \\ 0 & Q_2 \otimes W_2(a) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & Q_n \otimes W_n(a) \end{pmatrix}. \tag{4.12}$$

Recalling (4.9) we see that the action of each block of (4.12) is by the representation $C_\lambda \otimes W_\lambda$. □

Examples.

1. Let G be a group and let V and W be two representations of G. Define an action of G on the vector space $V \otimes W$ by

$$g(vw) = (gv)(gw),$$

for all $g \in G$, $v \in V$ and $w \in W$. The resulting representation of G is the *Kronecker product* $V \otimes_d W$ of the representations V and W (see also §5 Ex. 4). In matrix form, the representation $V \otimes W$ is given by setting

$$(V \otimes_d W)(g) = V(g) \otimes W(g),$$

for each $g \in G$. Note, however, that if we extend this action to an action of $A = \mathbb{C}G$ on $V \otimes W$, then for a general $a \in A$, $a(vw)$ is *not* equal to $(av)(aw)$and $(V \otimes_d W)(a)$ is *not* equal to $V(a) \otimes W(a)$.

2. Theorem (4.6) gives that there is a one-to-one correspondence between minimal central idempotents z_λ^C of C and characters χ_A^λ of irreducible representations of A appearing in the decomposition of V. Let Let χ_C^λ be the irreducible characters of C and for each λ set $d_\lambda^C = \chi_C^\lambda(1)$, so that the d_λ are the dimensions of the irreducible representations of C. The **Frobenius map** is the map

$$\begin{array}{rccc} F: & Z(C) & \longrightarrow & R(A) \\ & (1/d_\lambda^C)z_\lambda^C & \longmapsto & \chi_A^\lambda. \end{array}$$

Let $t: C\otimes A \to \mathbb{C}$ be the trace of the action of $C \otimes A$ on the representation V. By taking traces on each side of the isomorphism in Theorem (4.11) we have that

$$t(q,a) = \sum_\lambda \chi_C^\lambda(q)\chi_A^\lambda(a). \tag{4.13}$$

Let $\vec{t}_C = (t_\lambda^C)$ be a nondegenerate trace on C, let B be a basis of C and for each $g \in B$ let g^* be the element of the dual basis to B with respect to the trace $\vec{t}_C$ such that $\vec{t}_C(gg^*) = 1$. Then, for any $z \in Z(C)$, the center of C,

$$F(z) = \sum_{g\in B} \vec{t}_C(zg^*)t(g,\cdot), \tag{4.14}$$

since, using (3.8) and (3.9),

$$\begin{aligned} F(z_\mu^C/d_\mu^C) &= \sum_g (1/d_\mu^C)\vec{t}_C(z_\mu^C g^*)t(g,\cdot) \\ &= \sum_g (t_\mu^C/d_\mu^C)\chi_C^\mu(g^*)t(g,\cdot) \\ &= \sum_g (t_\mu^C/d_\mu^C)\chi_C^\mu(g^*)\sum_\lambda \chi_C^\lambda(g)\chi_A^\lambda(\cdot) \\ &= \sum_\lambda (t_\mu^C/d_\mu^C)\delta_{\mu\lambda}(d_\lambda^C/t_\lambda^C)\chi_\lambda^A(\cdot) \\ &= \chi_\mu^A(\cdot). \end{aligned}$$

If we apply the inverse F^{-1} of the Frobenius map to (4.13) we get

$$F^{-1}(t(q,\cdot)) = \sum_\lambda \chi_C^\lambda(q)(z_\lambda^C/d_\lambda^C).$$

Formula (3.13) shows that

$$F^{-1}(t(q,\cdot)) = (\sum_\lambda (t_\lambda^C/d_\lambda^C)z_\lambda^C)[q].$$

In the case that $\vec{t}_C$ is the trace of the regular representation $\sum_\lambda (t_\lambda^C/d_\lambda^C)z_\lambda^C = 1$ and $F^{-1}(t(q,\cdot)) = [q]$.

Notes and References

"Double centralizer nonsense" is a term that has been used by R. Stanley in reference to Theorems (4.6) and (4.12). I have chosen to adopt this term as well. These results are originally due to I. Schur [Sc1],[Sc2], and are often referred to as the Double Commutant Theorem, or, in the special case of the representation $V^{\otimes f}, \dim V = n$ of $Gl(n)$, Schur-Weyl duality. This was the key concept in Schur's original work on the rational representations of $Gl(n)$.

The Frobenius map given in Ex. 3 is a generalization of the classical Frobenius map [Mac] §I.7. In a paper [Fr] that demonstrates absolute genius, Frobenius used it as a tool for determining the characters of the symmetric groups.

5. Induction and Restriction.

Let A be a subalgebra of an algebra B.
Let V be a representation of B. The *restriction* $V\downarrow_A^B$ of V to A to be the representation of A given by the action of A on V. Let W be a representation of A. Define $B\otimes_A W$ to be all formal linear combinations of elements $b\otimes w$, where $b\in B, w\in W$ with the relations

$$\begin{aligned}(b_1+b_2)\otimes w &= (b_1\otimes w)+(b_2+w),\\ b\otimes(w_1+w_2) &= (b\otimes w_1)+(b\otimes w_2),\\ (\alpha b)\otimes w &= b\otimes(\alpha w)=\alpha(b\otimes w),\\ ba\otimes w &= b\otimes aw,\end{aligned}\tag{5.1}$$

for all $a\in A, b, b_1, b_2\in B, w, w_1, w_2\in W$ and $\alpha\in\mathbb{C}$. The *induced* representation $W\uparrow_A^B$ is the representation of B on $B\otimes_A W$ given by the action

$$b(b'\otimes w)=(bb')\otimes w, \tag{5.2}$$

for all $b, b'\in B$ and $w\in W$.

(5.3) Proposition. Let $A\subset B\subset C$ be such that A is a subalgebra of B and B is a subalgebra of C. Let V, V_1, V_2 be representations of C and let W, W_1, W_2 be representations of C.

1) $(V_1\oplus V_2)\downarrow_A^C\cong V_1\downarrow_A^C\oplus V_2\downarrow_A^C$.
2) $(V\downarrow_B^C)\downarrow_A^B\cong V\downarrow_A^C$.
3) $(V_1\oplus V_2)\uparrow_A^B= V_1\uparrow_A^B\oplus V_2\uparrow_A^B$.
4) $(V\uparrow_A^B)\uparrow_B^C\cong V\uparrow_A^C$.

Proof. 1) and 2) are trivial consequences of the definition. The fact that the map

$$\begin{array}{rccc}\phi: & B\otimes_A(V_1\oplus V_2) & \to & (B\otimes_A V_1)\oplus(B\otimes_A V_2)\\ & b\otimes(v_1,v_2) & \mapsto & (b\otimes v_1, b\otimes v_2).\end{array}$$

is a B-module isomorphism gives 3). The map

$$\begin{array}{rccc}\phi_1: & C\otimes_B(B\otimes_A V) & \to & (C\otimes_B B)\otimes_A V\\ & c\otimes(b\otimes v) & \mapsto & (c\otimes b)\otimes v\end{array}$$

and the map

$$\begin{array}{rccc}\phi_2: & C\otimes_B B & \to & C\\ & c\otimes b & \mapsto & cb\end{array}$$

are both C-module isomorphisms. So

$$C\otimes_B(B\otimes_A V)\cong(C\otimes_B B)\otimes_A V\cong C\otimes_A V,$$

giving 4). □

Note: Proving that these maps are isomorphisms is not a complete triviality. One must show that they are well defined (by showing that they preserve the bilinearity relations (5.1)) and that the inverse maps are also well defined. It is helpful to use the fact that the tensor product is a universal object as given in Ex. 1.

(5.4) Theorem. (Frobenius reciprocity) Let $A\subset B$ be algebras and V_λ and W_μ be irreducible representations of A and B respectively. Then

$$\mathrm{Hom}_B(V_\lambda\uparrow_A^B, W_\mu)\cong\mathrm{Hom}_A(V_\lambda, W_\mu\downarrow_A^B).$$

Proof. The map

$$\begin{array}{rccc}\Psi: & \mathrm{Hom}_B(B\otimes_A V_\lambda, W_\mu) & \to & \mathrm{Hom}_A(V_\lambda, W_\mu\downarrow_A^B)\\ & \phi & \mapsto & \phi'\end{array},$$

where

$$\phi'(v)=\phi(1\otimes v),$$

is an isomorphism. The inverse map is given by $\Psi^{-1}(\phi')=\phi$ where ϕ is given by

$$\phi(b\otimes v)=b\phi(1\otimes v)=b\phi'(v),$$

so that ϕ is a B-module homomorphism. □

Branching rules

Now suppose that A is a subalgebra of B and that both A and B are semisimple. Let $\hat{A}$ and $\hat{B}$ be index sets for the irreducible representations of A and B respectively. Let V_λ and W_μ be the irreducible representations of A and B labelled by $\lambda \in \hat{A}$ and $\mu \in \hat{B}$ respectively. Let $g_{\lambda\mu} \in \mathbb{Z}$ be such that

$$V_\lambda \uparrow_A^B \cong \oplus_{\mu \in \hat{B}} g_{\lambda\mu} W_\mu \tag{5.5}$$

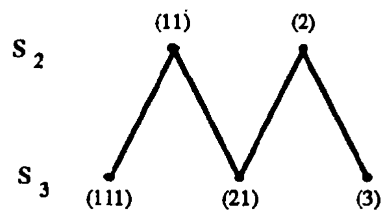
for each pair $(\lambda, \mu), \lambda \in \hat{A}, \mu \in \hat{B}$. Frobenius reciprocity implies that

$$W_\mu \downarrow_A^B \cong \oplus_{\lambda \in \hat{A}} g_{\lambda\mu} V_\lambda \tag{5.5$'$}$$

for each $\mu \in \hat{B}$. An equation of the form (5.5) or (5.5$'$) is called a *branching rule* between A and B.

One can produce a visual representation of branching rules in the form of a graph. Construct a graph with two rows of vertices, the vertices in the first row labelled by the elements of $\hat{A}$ and the vertices of the second row labelled by the elements of $\hat{B}$ such that the vertex labelled by $\lambda \in \hat{A}$ and the vertex labelled by $\mu \in \hat{B}$ are connected by $g_{\lambda\mu}$ edges. This graph is the *Bratteli diagram* of $A \subset B$.

As an example, the following diagram is the Bratteli diagram of $\mathbb{C}S_2 \subset \mathbb{C}S_3$, where S_n denotes the symmetric group. Recall that the irreducible representations of S_2 and S_3 are indexed by partitions of 2 and of 3 respectively.



Note that in this example each $g_{\lambda\mu}$ is either 0 or 1; there are no multiple edges.

Let $p \in A$ and consider the representation of A given by left multiplication on the space Aa. Then

$$(Ap) \uparrow_A^B \cong Bp. \tag{5.6}$$

To see this, informally, one notes that since $Ap \subset A$ we can move Ap across the tensor product to give,

$$(Ap) \uparrow_A^B = B \otimes_A Ap = BAp \otimes_A 1 = Bp \otimes_A 1 \cong Bp.$$

$BAp = Bp$ since $1 \in A$. More formally we should show that the map

$$\begin{array}{rcl} B \otimes_A Ap & \longrightarrow & Bp \\ b \otimes ap & \longmapsto & bap \end{array}$$

is well defined and has well defined inverse given by

$$b \otimes p \longleftarrow bp.$$

Now let p_λ be a minimal idempotent of A such that the action of A by left multiplication on Ap_λ is a representation of A isomorphic to the irreducible representation V_λ of A (3.6). Suppose that

$$p_\lambda = \sum q_i$$

is a decomposition (§1 Ex. 7) of the minimal idempotent p_λ of A into minimal orthogonal idempotents of B. Then $Bp_\lambda = B \sum q_i = \sum Bq_i$ gives a decomposition of Bp_λ into irreducible representations. So, by (5.6) and the branching rule (5.5), for exactly $g_{\lambda\mu}$ of the q_i we will have that Bq_i is isomorphic to the irreducible representation W_μ of B. We can write the decomposition of p_λ as

$$p_\lambda = \sum_{\mu \in \hat{B}} \sum_{i=1}^{g_{\lambda\mu}} q_{\mu i} \tag{5.7}$$

where each $q_{\mu i}$ is such that $Bq_{\mu i}$ is isomorphic to the irreducible representation W_μ of B.

Characters of induced representations

Let V be a representation of A where A is a subalgebra of an algebra B and both A and B are semisimple. Let χ_V be the character of V and let $\chi_{V\uparrow_B^A}$ be the character of $V\uparrow_A^B$. For each $a \in \mathcal{A}$ let a^* denote the element of the dual basis to $\mathcal{A}$ with respect to the trace, tr, of the regular representation of A such that $tr(aa^*) = 1$.

Let $\mathcal{B}$ be a basis of B and let $\vec{t}_B = (t_\mu^B)$ be a nondegenerate trace on B. For each $b \in \mathcal{B}$ let b^* denote the element of the dual basis to $\mathcal{B}$ with respect to the trace $\vec{t}_B$ such that $\vec{t}(bb^*) = 1$. For any element $x \in B$ we set (as in §3 Ex. 7)

$$[x] = \sum_{b\in\mathcal{B}} bxb^*.$$

(5.8) Theorem.

$$\chi_{V\uparrow_A^B}(b) = \sum_a \chi_V(a) < [b], a^* >,$$

where $< b_1, b_2 >= \vec{t}_B(b_1 b_2)$.

Proof. In keeping with the notations of earlier sections, let $\hat{A}$ and $\hat{B}$ be index sets for the irreducible representations of A and B respectively and let $\chi_A^\lambda, \lambda \in \hat{A}$ and $\chi_B^\mu, \mu \in \hat{B}$ denote the irreducible characters of A and B respectively. Let $z_\lambda^A, \lambda \in \hat{A}$ and $z_\mu^B, \mu \in \hat{B}$ denote the minimal central idempotents of A and B respectively. Let $d_\lambda^A = \chi_A^\lambda(1)$ so that d_λ is the dimension of the irreducible representation of A corresponding to $\lambda \in \hat{A}$.

We have the following facts:

1) (Theorem (3.10)) For each $\lambda \in \hat{A}$, $\mu \in \hat{B}$,

$$z_\lambda^A = \sum_{a\in\mathcal{A}} t_\lambda^A \chi_A^\lambda(a) a^*, \qquad \text{and}$$
$$z_\mu^B = \sum_{b\in\mathcal{B}} t_\mu^B \chi_B^\mu(b) b^*,$$

respectively.

2) (§3 Ex. 5) The trace vector (t_λ^A) of the trace of the regular representation of A is given by $t_\lambda^A = d_\lambda^A$ for all $\lambda \in \hat{A}$.

3) Suppose that $V \cong \oplus_{\lambda\in\hat{A}} V_\lambda^{\oplus m_\lambda}$ gives the decomposition of V into irreducible representations of A. Then

$$\chi_V(a) = \sum_{\lambda\in\hat{A}} m_\lambda \chi_A^\lambda(a),$$

for all $a \in A$.

4) The branching rule (5.5) for $A \subset B$ gives that

$$\chi_{V\uparrow_A^B}(b) = \sum_{\lambda\in\hat{A}} m_\lambda \sum_{\mu\in\hat{B}} g_{\lambda\mu} \chi_B^\mu(b),$$

for all $b \in B$.

5) For each $\lambda \in \hat{A}$ let

$$z_\lambda^A = \sum_{i=1}^{d_\lambda^A} p_{\lambda i}^A$$

be a decomposition of z_λ^A into minimal orthogonal idempotents of A. For each $\lambda \in \hat{A}$ and $1 \le i \le d_\lambda^A$ let

$$p_{\lambda i}^A = \sum_{\mu\in\hat{B}} \sum_{j=1}^{g_{\lambda\mu}} q_{\mu j}^B$$

be a decomposition (5.7) of $p^A_{\lambda i}$ into minimal orthogonal idempotents of B. $q_{\mu j}$ denotes a minimal idempotent in the minimal ideal of B corresponding to $\mu \in \hat{b}$, i.e., a minimal idempotent such that the representation $Bq_\mu j$ of B is isomorphic to the ireeducible representation of B corresponding to $\mu \in \hat{B}$. Then, by (3.12),

$$[q^B_{\mu j}] = (1/t^B_\mu) z^B_\mu ,$$

for each minimal idempotent $q^B_{\mu j}$, since for each $\nu \in \hat{B}$, $\chi_\nu(q^B_{\mu j}) = \delta_{\mu\nu}$.

6) Let $b_1, b_2 \in B$. Using the trace property,

$$\begin{aligned} < [b_1], b_2 > &= \vec{t}_B(\sum_{b \in B} b b_1 b^* b_2) \\ &= \vec{t}_B(\sum_{b \in B} b_1 b^* b_2 b) \\ &=< b_1, [b_2] > . \end{aligned}$$

Now, define

$$z = \sum_{\lambda \in \hat{A}} (m_\lambda / d^A_\lambda) z^A_\lambda .$$

Then, using 1), 2) and 3),

$$\begin{aligned} z &= \sum_{\lambda \in \hat{A}} m_\lambda \sum_a (t^A_\lambda / d^A_\lambda) \chi^\lambda_A(a) a^* \\ &= \sum_a \chi_V(a) a^*, \end{aligned}$$

and, by 5), 1) and 4),

$$\begin{aligned} [z] &= \sum_\lambda (m_\lambda / d^A_\lambda) [z^A_\lambda] \\ &= \sum_\lambda (m_\lambda / d^A_\lambda) \sum_{i=1}^{d^A_\lambda} [p^A_{\lambda i}] \\ &= \sum_\lambda (m_\lambda / d^A_\lambda) \sum_{i=1}^{d^A_\lambda} \sum_\mu \sum_{j=1}^{g_{\lambda\mu}} [q^B_{\mu j}] \\ &= \sum_\lambda (m_\lambda / d^A_\lambda) \sum_{i=1}^{d^A_\lambda} \sum_\mu \sum_{j=1}^{g_{\lambda\mu}} (1/t^B_\mu) z^B_\mu \\ &= \sum_\lambda \sum_\mu (m_\lambda / d^A_\lambda) d^A_\lambda g_{\lambda\mu} (1/t^B_\mu) \sum_b t^B_\mu \chi^\mu_B(b) b^* \\ &= \sum_b \chi_{V\uparrow^B_A}(b) b^* . \end{aligned}$$

Combining these and using 6) we get

$$\begin{aligned} \chi_{V\uparrow^B_A}(b) &=< [z], b > \\ &=< [\sum_a \chi_V(a) a^*], b > \\ &= \sum_a \chi_V(a) < [a^*], b > \\ &= \sum_a \chi_V(a) < a^*, [b] >, \end{aligned}$$

as desired. □

Centralizers

Let A be a subalgebra of an algebra B, and let V be a representation of B. Let $\overline{A}$ and $\overline{B}$ be the centralizers of $V(A)$ and $V(B)$ respectively. Then $\overline{B}$ is a subalgebra of $\overline{A}$; $A \subset B$ and $\overline{A} \supset \overline{B}$.

(5.9) Theorem. Suppose that

$$W_\mu \downarrow_A^B \cong \sum_\lambda g_{\mu\lambda} V_\lambda,$$

$$\overline{V}_\lambda \downarrow_{\overline{B}}^{\overline{A}} \cong \sum_\mu g'_{\lambda\mu} \overline{W}_\mu$$

are the branching rules for $A \subset B$ and $\overline{B} \subset \overline{A}$ respectively. Then for all λ, μ

$$g_{\mu\lambda} = g'_{\lambda\mu}.$$

Proof. We know, Theorem (4.11), that, as $A \otimes \overline{A}$ representations,

$$V \cong \oplus_\lambda V_\lambda \otimes \overline{V}_\lambda,$$

and as $B \otimes \overline{B}$ representations,

$$V \cong \oplus_\mu W_\mu \otimes \overline{W}_\mu,$$

where $V_\lambda, \overline{V}_\lambda, W_\mu$, and $\overline{W}_\mu$ are irreducible representations of $A, \overline{A}, B$, and $\overline{B}$ respectively.

$A \otimes \overline{B}$ is a subalgebra of both $A \otimes \overline{A}$ and $B \otimes \overline{B}$. We have that as $A \otimes \overline{B}$ representations

$$\begin{aligned} V \cong V \downarrow_{A\otimes\overline{B}}^{A\otimes\overline{A}} &\cong \oplus_\lambda V_\lambda \otimes (\oplus_\mu g'_{\lambda\mu} \overline{W}_\mu) \\ &\cong \oplus_{\lambda,\mu} g'_{\lambda\mu} V_\lambda \otimes \overline{W}_\mu. \end{aligned}$$

On the other hand as $A \otimes \overline{B}$ representations

$$\begin{aligned} V \cong V \downarrow_{A\otimes\overline{B}}^{B\otimes\overline{B}} &\cong \oplus_\mu (\oplus_\lambda g_{\mu\lambda} V_\lambda) \otimes \overline{W}_\mu \\ &\cong \oplus_{\lambda,\mu} g_{\mu\lambda} V_\lambda \otimes \overline{W}_\mu. \quad \square \end{aligned}$$

Examples.

1. Let A, B and C be vector spaces. A map $f: A \times B \to C$ is bilinear if

$$\begin{aligned} f(a_1 + a_2, b) &= f(a_1, b) + f(a_2, b), \\ f(a, b_1 + b_2) &= f(a, b_1) + f(a, b_2), \\ f(\alpha a, b) = f(a, \alpha b) &= \alpha f(a, b), \end{aligned}$$

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $\alpha \in \mathbb{C}$.

The tensor product is given by a vector space $A \otimes B$ and a map $i: A \times B \to A \otimes B$ such that for every bilinear map $f: A \times B \to C$ there exists a linear map $\bar{f}: A \otimes B \to C$ such that the following diagram commutes.

$$\begin{tikzcd} A \times B \arrow[d, "i"] \arrow[dr, "f"] & \\ A \otimes B \arrow[r, "\bar{f}"'] & C. \end{tikzcd}$$

One constructs the tensor product $A \otimes B$ as the vector space of elements $a \otimes b$, $a \in A$, $b \in B$, with relations

$$\begin{aligned} (a_1 + a_2) \otimes b &= a_1 \otimes b + a_2 \otimes b, \\ a \otimes (b_1 + b_2) &= a \otimes b_1 + a \otimes b_2, \\ (\alpha a) \otimes b = a \otimes (\alpha b) &= \alpha (a \otimes b), \end{aligned}$$

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $\alpha \in \mathbb{C}$. The map $i: A \times B \to A \otimes B$ is given by $i(a,b) = a \otimes b$. Using the above universal mapping property one gets easily that the tensor product is unique in the sense that any two tensor products of A and B are isomorphic.

If R is an algebra and A is a right R-module (a vector space that affords an antirepresentation of R) and B a left R-module then one forms the vector space $A \otimes_R B$ as above except that we require a bilinear map $f: A \times B \to C$ to satisfy the additional condition

$$f(ar, b) = f(a, rb)$$

for all $r \in R$. Then the tensor product $A \otimes_R B$ is a vector space that satisfies the universal mapping property given above. To construct $A \otimes_R B$ one again uses the vector space of elements $a \otimes b$, $a \in A$, $b \in B$, with the relations above and the additional relation

$$ar \otimes b = a \otimes rb,$$

for all $r \in R$.

2. Let $A \subset B$ be semisimple algebras such that A is a subalgebra of B. Let $\hat{A}$ and $\hat{B}$ be index sets for the irreducible representations of A and B respectively, and suppose that $\{f^\mu_{ij}\}$, $\mu \in \hat{A}$, is a complete set of matrix units of A.

(5.10) Theorem. [Bt] There exists a complete set of matrix units $\{e^\lambda_{rs}\}, \lambda \in \hat{B}$, of B that is a refinement of the f^μ_{ij} in the sense that for each $\mu \in \hat{A}$ and each i,

$$f^\mu_{ii} = \sum e^\lambda_{rr},$$

for some set of e^λ_{rr}.

Proof. Suppose that $B \cong \oplus_{\lambda \in \hat{B}} M_{d_\lambda}(\mathbb{C})$. Let z^B_λ be the minimal central idempotent of B such that $I_\lambda = Bz_\lambda$ is the minimal ideal corresponding to the λ block of matrices in $\oplus_\lambda M_{d_\lambda}(\mathbb{C})$.

For each $\mu \in \hat{A}$ and each i decompose f^μ_{ii} into minimal orthogonal idempotents of B (§1 Ex. 7), $f^\mu_{ii} = \sum p_j$. Label each p_j appearing in this sum by the element $\lambda \in \hat{B}$ which indexes the minimal ideal $I_\lambda = Bp_jB$ of B. Then

$$1 = \sum_{\mu,i} f^\mu_{ii} = \sum_{\lambda \in \hat{B}} \sum_{j=1}^{d_\lambda} p^\lambda_j.$$

Now,

$$B = 1 \cdot B \cdot 1 = \sum_{\lambda,\mu \in \hat{B}} \sum_{\substack{1 \le i \le d_\lambda \\ 1 \le j \le d_\mu}} p^\lambda_i B p^\mu_j.$$

If $\lambda \neq \mu$ then the space $p^\lambda_i B p^\mu_j = p^\lambda_i B(z^B_\mu p^\mu_j) = p^\lambda_i z^B_\mu B p^\mu_j = 0$ for all i, j. Since $p^\lambda_i = p^\lambda_i \cdot 1 \cdot p^\lambda_i \in p^\lambda_i I_\lambda p^\lambda_i$ and $p^\lambda_i B p^\lambda_j p^\lambda_j B p^\lambda_i = p^\lambda_i I_\lambda p^\lambda_i \neq 0$, we know that $p^\lambda_i B p^\lambda_j$ is not zero for any $1 \le i, j \le d_\lambda$. Futhermore, since the dimension of B is $\sum_\lambda d^2_\lambda$ each of the spaces $p^\lambda_i B p^\lambda_j$ is one dimensional.

For each p^λ_i define $e^\lambda_{ii} = p^\lambda_i$. For each λ and each $1 \le i < j \le d_\lambda$ let e^λ_{ij} be some element of $p^\lambda_i B p^\lambda_j$. Then choose $e^\lambda_{ji} \in p^\lambda_j B p^\lambda_i$ such that $e^\lambda_{ij} e^\lambda_{ji} = e^\lambda_{ii}$. This defines a complete set of matrix units of B. □

3. Let G be a finite group and let H be a subgroup of G. Let $R = \{g_i\}$ be a set of representatives for the left cosets gH of H in G. The action of G on the cosets of H in G by left multiplication defines a representation π_H of G. This representation is a *permutation representation* of G. Let $g \in G$. The entries $\pi_H(g)_{i'i}$ of the matrix $\pi_H(g)$ are given by $\pi_H(g)_{i'i} = \delta_{i'k}$ where k is such that $gg_i \in g_kH$.

Let V be a representation of H. Let $B = \{v_j\}$ be a basis of V. Then the elements $g \otimes v_j$ where $g \in G, v_j \in B$ span $\mathbb{C}G \otimes_{\mathbb{C}H} V$. The fourth relation in (5.1) gives that the set $\{g_i \otimes v_j\}$, $g_i \in R$, $v_j \in B$ forms a basis of $\mathbb{C}G \otimes_{\mathbb{C}H} V$.

Let $g \in G$ and suppose that $gg_i = g_k h$, where $h \in H$ and $g_k \in R$. Then

$$\begin{aligned} gg_i \otimes v_j &= g_k h \otimes v_j \\ &= g_k \otimes hv_j \\ &= \sum_{j'} g_k \otimes v_{j'} V(h)_{j'j} \\ &= \sum_{i',j'} g_{i'} \otimes v_{j'} V(h)_{j'j} \delta_{i'k} \\ &= \sum_{i',j'} g_{i'} \otimes v_{j'} V(h)_{j'j} \pi_H(g)_{i'i}. \end{aligned}$$

Then

$$\begin{aligned} \chi_{V\uparrow_H^G}(g) &= \sum_{g_i \in R, v_j \in B} gg_i \otimes v_j |_{g_i \otimes v_j} \\ &= \sum_{\substack{g_i, v_j \\ gg_i \in g_i H}} V(g_i^{-1} g g_i)_{jj}. \end{aligned}$$

Since characters are constant on conjugacy classes we have that

$$\begin{aligned} \chi_{V\uparrow_H^G}(g) &= (1/|H|) \sum_{h \in H} \sum_{\substack{g_i \\ h^{-1} g_i^{-1} g g_i h \in H}} \chi_V(h^{-1} g_i^{-1} g g_i h) \\ &= (1/|H|) \sum_{\substack{a \in H \\ a \in C_g}} \chi_V(a), \end{aligned}$$

where C_g denotes the conjugacy class of g. This is an alternate proof of Theorem (5.8) for the special case of inducing from a subgroup H of a group G to the group G.

4. Define $\mathbb{C}G \otimes_d \mathbb{C}G$ to be the subalgebra of the algebra $\mathbb{C}G \otimes \mathbb{C}G$ consisting of the span of the elements $g \otimes g$, $g \in G$. Then $\mathbb{C}G \cong \mathbb{C}G \otimes_d \mathbb{C}G$ as algebras.

Let V_1 and V_2 be representations of G. Then the restriction of the $\mathbb{C}G \otimes \mathbb{C}G$ representation $V = V_1 \otimes V_2$ to the algebra $\mathbb{C}G \otimes_d \mathbb{C}G$ is the Kronecker product (§4 Ex.1)

$$V_1 \otimes_d V_2 = V_1 \otimes V_2 \downarrow^{\mathbb{C}G \otimes \mathbb{C}G}_{\mathbb{C}G \otimes_d \mathbb{C}G}$$

of V_1 and V_2. Since $\mathbb{C}G \cong \mathbb{C}G \otimes_d \mathbb{C}G$ we can view $V_1 \otimes_d V_2$ as a representation of G.

Let V_λ and V_μ be irreducible representations of G Such that $V_\lambda \otimes V_\mu$ appears as an irreducible component of the $\mathbb{C}G \otimes \mathbb{C}G$ representation $V_1 \otimes V_2$. The decomposition of the Kronecker product

$$V_\lambda \otimes_d V_\mu = V_1 \otimes V_2 \downarrow^{\mathbb{C}G \otimes \mathbb{C}G}_{\mathbb{C}G \otimes_d \mathbb{C}G} \cong \oplus_\nu g^\nu_{\lambda\mu} V_\nu$$

into irreducible representations V_ν of G is given by the branching rule for $\mathbb{C}G \otimes \mathbb{C}G \supset \mathbb{C}G \otimes_d \mathbb{C}G$. Let C_1 and C_2 be the centralizers of the representations V_1 and V_2 respectively. Let C be the centralizer of the $\mathbb{C}G \otimes \mathbb{C}G$ representation $V = V_1 \otimes V_2$. Applying Theorem (5.9) to V where $A = \mathbb{C}G \otimes \mathbb{C}G$ and $B = \mathbb{C}G \otimes_d \mathbb{C}G \cong G$ shows that the $g^\nu_{\lambda\mu}$ are also given by the branching rule for $C_1 \otimes C_2 \subset C$.

Notes and References

The main result, Theorem (5.8), of this section is a generalization of the formula for the induced character for finite groups, see [Se] §7.2. I have been unable to find any similar result in previous literature.

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