Symmetric functions
Lecture Notes

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1. Symmetric functions

The symmetric group S_n acts on the vector space

$$\mathbb{Z}^n = \mathbb{Z}\text{-span}\{x_1, \ldots, x_n\} \qquad \text{by} \qquad wx_i = x_{w(i)},$$

for $w \in S_n$, $1 \le i \le n$. This action induces an action of S_n on the polynomial ring $\mathbb{Z}[X_n] = \mathbb{Z}[x_1, \ldots, x_n]$ by ring automorphisms. For a sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ of nonnegative integers let

$$x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}, \qquad \text{so that} \quad \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}\text{-span}\{x^\gamma \mid \gamma \in \mathbb{Z}_{\ge 0}^n\}.$$

The ring of *symmetric functions* is

$$\mathbb{Z}[X_n]^{S_n} = \{f \in \mathbb{Z}[X_n] \mid wf = f \text{ for all } w \in S_n\}, \tag{1.1}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma, \qquad \text{for } \lambda \in \mathbb{Z}_{\ge 0}^n,$$

where $S_n\lambda$ is the orbit of λ under the action of S_n. Let

$$P^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\ge 0}^n \mid \lambda_1 \ge \cdots \ge \lambda_n\} \tag{1.2}$$

so that

$$\{m_\lambda \mid \lambda \in P^+\} \quad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[X_n]^{S_n}. \tag{1.3}$$

Partitions

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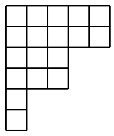
A *partition* is a collection μ of boxes in a corner where the convention is that gravity goes up and to the left. As for matrices, the rows and columns of μ are indexed from top to bottom and left to right, respectively.

$$\begin{array}{lll} \text{The } \textit{parts} \text{ of } \mu \text{ are} & \mu_i = (\text{the number of boxes in row } i \text{ of } \mu), & \\ \text{the } \textit{length} \text{ of } \mu \text{ is} & \ell(\mu) = (\text{the number of rows of } \mu), & (1.4) \\ \text{the } \textit{size} \text{ of } \mu \text{ is} & |\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)} = (\text{the number of boxes of } \mu). & \end{array}$$

Then μ is determined by (and identified with) the sequence $\mu = (\mu_1, \ldots, \mu_\ell)$ of positive integers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$, where $\ell = \ell(\mu)$. For example,

$$(5, 5, 3, 3, 1, 1) =$$  .

A *partition of* k is a partition λ with k boxes. Write $\lambda \vdash k$ if λ is a partition of k. Make the convention that $\lambda_i = 0$ if $i > \ell(\lambda)$. The *dominance order* is the partial order on the set of partitions of k,

$$P^+(k) = \{\text{partitions of } k\} = \{\lambda = (\lambda_1, \ldots, \lambda_\ell) \mid \lambda_1 \geq \cdots \geq \lambda_\ell > 0,\ \lambda_1 + \ldots + \lambda_\ell = k\},$$

given by

$$\lambda \geq \mu \qquad \text{if} \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for all } 1 \leq i \leq \max\{\ell(\lambda), \ell(\mu)\}.$$

PUT THE PICTURE OF THE HASSE DIAGRAM FOR $k = 6$ HERE.

Tableaux

Let λ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ be a sequence of nonnegative integers. A *column strict tableau of shape* λ *and weight* μ is a filling of the boxes of λ with μ_1 1s, μ_2 2s, ..., μ_n ns, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If p is a column strict tableau write $\text{shp}(p)$ and $\text{wt}(p)$ for the shape and the weight of p so that

$$\begin{array}{rll} \text{shp}(p) = (\lambda_1, \ldots, \lambda_n), & \text{where} & \lambda_i = \text{number of boxes in row } i \text{ of } p, \quad \text{and} \\ \text{wt}(p) = (\mu_1, \ldots, \mu_n), & \text{where} & \mu_i = \text{number of } i\text{s in } p. \end{array}$$

For example,

$$p = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & 3 & 3 & 4 \\ \cline{1-7} 3 & 3 & 3 & 4 & 4 & 4 & 5 \\ \cline{1-7} 4 & 5 & 5 & 6 \\ \cline{1-4} 6 & 7 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \text{has} \quad \begin{array}{l} \text{shp}(p) = (9, 7, 7, 4, 2, 1, 0) \quad \text{and} \\ \text{wt}(p) = (7, 6, 5, 5, 3, 2, 2). \end{array}$$

For a partition λ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } p \mid \text{shp}(p) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } p \mid \text{shp}(p) = \lambda \text{ and } \text{wt}(p) = \mu\}, \end{aligned} \tag{1.5}$$

Elementary symmetric functions

Define symmetric functions e_r, $0 \leq r \leq n$, via the generating function

$$\prod_{i=1}^{n}(1 - x_i z) = \sum_{r=0}^{n}(-1)^r e_r z^r.$$

Then $e_0 = 1$ and, for $0 \leq r \leq n$,

$$e_r = m_{(1^r)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} = \sum_{\text{shp}(p) = (1^r)} x^{\text{wt}(p)},$$

where the last sum is over all column strict tableaux p of shape (1^r).

If $f(t)$ is a polyomial in t with roots $\gamma_1, \ldots, \gamma_n$ then

$$\text{the coefficient of } t^r \text{ in } f(t) \text{ is } (-1)^{n-r} e_r(\gamma_1, \ldots, \gamma_n).$$

If A is an $n \times n$ matrix with entries in $\mathbb{F}$ with eigenvalues $\gamma_1, \ldots, \gamma_n$ then the trace of the action of A on the r^{th} exterior power of the vector space $\mathbb{F}^n$ is

$$\text{tr}(A, \textstyle\bigwedge^r \mathbb{F}^n) = e_r(\gamma_1, \ldots, \gamma_n), \quad \text{so that} \quad \text{Tr}(A) = e_1(\gamma_1, \ldots, \gamma_n), \quad \det(A) = e_n(\gamma_1, \ldots, \gamma_n),$$

and the characteristic polynomials of A is

$$\text{char}_t(A) = \sum_{r=0}^{n}(-1)^{n-r} e_{n-r}(\gamma_1, \ldots, \gamma_n) t^r.$$

Proposition 1.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. Then*

$$e_{\lambda'} = \sum_{\mu \leq \lambda} a_{\lambda'\mu} m_\mu,$$

where $a_{\lambda'\mu}$ is the number of matrices with entries from $\{0, 1\}$ with row sums λ' and column sums μ. Furthermore, $a_{\lambda'\lambda} = 1$ and $a_{\lambda'\mu} = 0$ unless $\mu \leq \lambda$.

Proof. If A is an $\ell \times n$ matrix with entries from $\{0, 1\}$ let

$$x^A = \prod_{i=1}^{n}(x_i)^{a_{ij}}$$

and define

$$rs(A) = (\rho_1, \ldots, \rho_n), \qquad cs(A) = (\gamma_1, \ldots, \gamma_n), \qquad \text{where} \qquad \rho_i = \sum_{j=1}^{\ell} a_{ij} \quad \text{and} \quad \gamma_j = \sum_{i=1}^{n} a_{ij},$$

so that $rs(A)$ and $cs(A)$ are the sequences of row sums and column sums of A, respectively. If $\lambda' = (\lambda'_1, \ldots, \lambda'_\ell)$ then

$$e_{\lambda'} = \prod_{j=1}^{\ell} e_{\lambda'_j} = \sum_{rs(A)=\lambda'} x^A = \sum_{\gamma \in \mathbb{Z}^n_{\geq 0}} \sum_{\substack{rs(A)=\lambda' \\ cs(A)=\gamma}} x^\gamma = \sum_\mu a_{\lambda'\mu} m_\mu.$$

Since there is a unique matrix A with $rs(A) = \lambda'$ and $cs(A) = \lambda$, $a_{\lambda'\lambda} = 1$. If A is a $0, 1$ matrix with $rs(A) = \lambda'$ and $cs(A) = mu$ then $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ since there are at most $\lambda_1 + \cdots + \lambda_i$ nonzero entries in the first i columns of A. Thus $a_{\lambda'\mu} = 0$ unless $\mu \leq \lambda$. ∎

Corollary 1.7.
(a) The set $\{e_\lambda \mid \ell(\lambda') \leq n\}$ is a basis of $\mathbb{Z}[X_n]^{S_n}$.
(b) $\mathbb{Z}[X_n]^{S_n} = \mathbb{Z}[e_1, \ldots, e_n]$.

Complete symmetric functions

Define symmetric functions h_r, $r \in \mathbb{Z}_{\geq 0}$, via the generating function

$$\prod_{i=1}^{n} \frac{1}{1 - x_i z} = \sum_{r \in \mathbb{Z}_{\geq 0}} h_r z^r.$$

Then $h_0 = 1$ and, for $r \in \mathbb{Z}_{>0}$,

$$h_r = \sum_{\lambda \vdash r} m_\lambda = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} = \sum_{\mathrm{sh}(p)=(r)} x^{\mathrm{wt}(p)},$$

where the last sum is over all column strict tableaux p of shape (r).

Proposition 1.8. *There is an involutive automorphism ω of $\mathbb{Z}[X_n]^{S_n}$ defined by*

$$\begin{array}{rccc} \omega\colon & \mathbb{Z}[X_n]^{S_n} & \longrightarrow & \mathbb{Z}[X_n]^{S_n} \\ & e_k & \longmapsto & h_k \end{array}$$

Proof. Comparing coefficients of z^k on each side of

$$1 = \left(\prod_{i=1}^{n} (1 - x_i z) \right) \left(\prod_{i=1}^{n} \frac{1}{1 - x_i z} \right) \qquad \text{yields} \qquad 0 = \sum_{r=1}^{k} (-1)^r e_r h_{n-r}.$$

∎

Corollary 1.9.
(a) The set $\{h_\lambda \mid \ell(\lambda') \le n\}$ *is a basis of* $\mathbb{Z}[X_n]^{S_n}$.
(b) $\mathbb{Z}[X_n]^{S_n} = \mathbb{Z}[h_1, \ldots, h_n]$.

Theorem 1.10. *The monomials in* $\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \mid 0 \le \epsilon_i \le n-i\}$ *form a basis of* $\mathbb{Z}[x_1, \ldots, x_n]$ *as an* $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ *module.*

Proof. Let $I = \langle e_1, \ldots, e_n \rangle$ be the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by $e_1, \ldots, e_n$. Since $(1 - x_1 t) \cdots (1 - x_n t) = 0 \bmod I$,

$$(1 - x_{i+1} t) \cdots (1 - x_n t) = \frac{1}{(1 - x_1 t) \cdots (1 - x_i t)} \mod I,$$

and so

$$\sum_{r=0}^{n-i} (-1)^r e_r(x_{i+1}, \ldots, x_n) t^r = \sum_{\ell \ge 0} h_\ell(x_1, \ldots, x_i) t^\ell \qquad \mod I.$$

Comparing coefficients of t^{n-i+1} on each side gives that, for all $1 \le i \le n$,

$$0 = h_{n-i+1}(x_1, \ldots, x_i) = \sum_{r=0}^{n-i+1} x_i^{n-i+1-r} h_r(x_1, \ldots, x_{i-1}) \mod I,$$

and thus

$$x_i^{n-i+1} = -\sum_{r=1}^{n-i+1} x_i^{n-i+1-r} h_r(x_1, \ldots, x_{i-1}) \mod I. \tag{1.11}$$

This identity shows (by induction on i) that x_i^{n-i+1} can be rewritten, mod I, as a linear combination of monomials in $x_1, \ldots, x_i$ with the exponent of x_i being $\le n - i$. In particular,

$$0 = h_{n-1+1}(x_1) = x_1^n \mod I$$

and it follows that any polynomial can be written, mod I, as a linear combination of monomials

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \qquad \text{with} \qquad 0 \le \epsilon_i \le n - i. \tag{1.12}$$

If S^k is the set of homogeneous degree k polynomials in $S = \mathbb{Z}[x_1, \ldots, x_n]$ and $(S^W)^k$ is the set of homogeneous degree k polynomials in $S^W = \mathbb{Z}[e_1, \ldots, e_n] = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ the the Poincaré series of S and S^W are

$$\frac{1}{(1-t)^n} = \sum_{k \ge 0} \dim(S^k) t^k \qquad \text{and} \qquad \prod_{i=1}^n \left(\frac{1}{1 - t^i} \right) = \sum_{k \ge 0} \dim((S^W)^k) t^k.$$

Then the Poincaré series of S/I is

$$\prod_{i=1}^n \frac{1 - t^i}{1 - t} = [n]! = 1 \cdot (1 + t) \cdots (1 + t + \cdots t^{n-1}).$$

There are $n(n-1)\cdots 2\cdot 1 = n!$ monomials in (???) and thus the monomials in (*) form a basis of S as an S^W module. The relations (???) provide a way to expand any polynomial in terms of this basis (with coefficients in S^W). ∎

2. The groups $G_{r,p,n}$

Let r and n be postive integers. The group $G_{r,1,n}$ is the group of $n\times n$ matrices with

(a) exactly one non zero entry in each row and each column,
(b) the nonzero entries are r^{th} roots of 1.

Let p be a positive integer (not necessarily prime) such that p divides r. The group $G_{r,p,n}$ is defined by the exact sequence

$$\{1\} \longrightarrow G_{r,p,n} \longrightarrow G_{r,1,n} \xrightarrow{\phi} \mathbb{Z}/p\mathbb{Z} \longrightarrow \{1\}\ , \qquad \text{where} \qquad \phi(g) = \left(\prod_{g_{ij}\neq 0} g_{ij}\right)^p$$

is the p^{th} power of the product of the nonzero entries of g, and $\mathbb{Z}/p\mathbb{Z}$ is identified with the group of p^{th} roots of unity. Thus $G_{r,p,n} = \ker\phi$ is a normal subgroup of $G_{r,1,n}$ of index p. Examples are

(a) $G_{1,1,n} = S_n = WA_{n-1}$ is the symmetric group (the Weyl group of type A_{n-1}),
(b) $G_{2,1,n} = O_n(\mathbb{Z}) = WB_n$ is the hyperoctahedral group of orthogonal matrices with entries in $\mathbb{Z}$ (the Weyl group of type B_n),
(c) $G_{2,2,n} = WD_n$ is the group of signed permutations with an even number of negative signs (the Weyl group of type D_n),
(d) $G_{r,1,1} = \mathbb{Z}/r\mathbb{Z}$ is the cyclic group of order r of r^{th} roots of unity, and
(e) $G_{r,r,2} = WI_2(r)$ is the dihedral group of order $2r$.

Let $\xi = e^{2\pi i/r}$ be a primitive rth root of unity and let $\mathfrak{o} = \mathbb{Z}[\xi]$. If $x_1,\ldots,x_n$ is a basis of $\mathfrak{o}^n$ then the natural action of $G_{r,p,n}$ extends uniquely to an action of $G_{r,p,n}$ on the polynomial ring $\mathfrak{o}[x_1,\ldots,x_n]$ by ring automorphisms. The *invariant ring* is

$$\mathfrak{o}[x_1,\ldots,x_n]^{G_{r,p,n}} = \{f\in\mathfrak{o}[x_1,\ldots,x_n] \mid wf = f \text{ for all } w\in G_{r,p,n}\}.$$

Proposition 2.1. *Let*

$$f_i(x_1,\ldots,x_n) = e_i(x_1^r,\ldots,x_n^r), \quad \textit{for } 1\le i\le n-1 \quad \textit{and}$$
$$f_n(x_1,\ldots,x_n) = e_n(x_1^{r/p},\ldots,x_n^{r/p}).$$

(a) $\mathfrak{o}[x_1,\ldots,x_n]^{G_{r,p,n}} = \mathfrak{o}[f_1,\ldots,f_n]$.
(b) $\mathfrak{o}[x_1,\ldots,x_n]$ *is a free* $\mathfrak{o}[x_1,\ldots,x_n]^{G_{r,p,n}}$*-module with basis*

$$\{x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n} \mid 0\le\epsilon_1\le r/p-1 \textit{ and } 0\le\epsilon_i\le ir-1, \textit{ for } 2\le i\le n\}.$$

Proof. To show: $f_1,\ldots,f_n$ generate $\mathfrak{o}[X_n]^W$ and they are algebraically independent. ∎

Each element $w \in G_{r,1,n}$ can be written uniquely in the form

$$w = t_1^{\gamma_1} \cdots t_n^{\gamma_n} \sigma, \qquad \text{where} \quad t_i = \mathrm{diag}(1, \ldots, 1, \xi, 1, \ldots, 1), \quad \sigma \in S_n, \quad 0 \leq \gamma_i \leq r-1,$$

so that t_i is the diagonal matrix with 1s on the diagonal except for ξ in the i^{th} diagonal entry. The element

$$w \in G_{r,p,n} \qquad \text{if} \qquad \gamma_1 + \cdots + \gamma_n = 0 \mod p,$$

and thus

$$G_{r,p,n} = \{w = t_1^{\gamma_1} \cdots t_n^{\gamma_n} \sigma \mid \sigma \in S_n,\ 0 \leq \gamma_n \leq r/p - 1, \text{ and } 0 \leq \gamma_i \leq r-1 \text{ for } 1 \leq i \leq n-1\}.$$

For each $w \in G_{r,p,n}$ define a monomial

$$x_w = \left(\prod_{j=1}^{n} (x_{\sigma(1)} \cdots x_{\sigma(j)})^{\gamma_j} \right) \left(\prod_{\substack{i \text{ suchthat} \\ \sigma(i) > \sigma(i+1)}} (x_{\sigma(1)} \cdots x_{\sigma(i)}) \right).$$

Proposition 2.2. *The polynomial ring $\mathfrak{o}[x_1, \ldots, x_n]$ is a free $\mathfrak{o}[x_1, \ldots, x_n]^{G_{r,p,n}}$-module with basis*

$$\{x_w \mid w \in G_{r,p,n}\}.$$

Proof. ∎

3. General W

Theorem 3.1. *Let V be a finite dimensional vector space over a field $\mathbb{F}$. Let W be a finite subgroup of $GL(V)$. If $S(V)^W$ is a polynomial algebra then W is generated by reflections.*

Proof. Let

$$I = \langle f \in S(V)^W \mid f(0) = 0 \rangle,$$

be the ideal in $S(V)$ generated by polynomials without constant term. Let $e_1, \ldots, e_r$ be homogeneous generators of I (which exist, by Hilbert).

Step 1. Every $f \in S(V)^W$ is a polynomial in $e_1, \ldots, e_r$.

Proof. The proof is by induction on the degree of f. Assume f is homogeneous and $\deg(f) > 0$. Since $f \in I$,

$$f = \sum_{i=1}^{r} p_i e_i, \qquad \text{with } p_i \in S(V),$$

and so

$$f = \frac{1}{|W|} \sum_{w \in W} wf = \sum_{i=1}^{r} \left(\frac{1}{|W|} \sum_{w \in W} wp_i \right) e_i,$$

and since the internal sum has lower degree it can be written as a polynomial in $e_1, \ldots, e_r$. ∎

Step 2. $r = \dim(V)$.

Proof. Let $n = \dim(V)$, let $x_1, \ldots, x_n$ be a basis of V and let $\mathbb{C}(x_1, \ldots, x_n)$ be the field of fractions of $S(V) = \mathbb{C}[x_1, \ldots, x_n]$. Since x_i is a root of

$$m_i(t) = \prod_{w \in W} (t - wx_i) \quad \in S(V)^W[t],$$

the variable x_i is algebraic over $\mathbb{C}(e_1, \ldots, e_r)$ the field of fractions of $S(V)^W$. Thus

$$0 = \operatorname{trdeg}\left(\frac{\mathbb{C}(x_1, \ldots, x_n}{\mathbb{C}(e_1, \ldots, e_r)}\right) = \operatorname{trdeg}\left(\frac{\mathbb{C}(x_1, \ldots, x_n)}{\mathbb{C}}\right) - \operatorname{trdeg}\left(\frac{\mathbb{C}(e_1, \ldots, e_r)}{\mathbb{C}}\right) = n - r. \qquad ∎$$

Step 3. The *Jacobian* of a map

$$\begin{array}{rccc} \varphi\colon & V & \longrightarrow & V \\ & x & \longmapsto & (\varphi_1(x), \ldots, \varphi_n(x)) \end{array} \qquad \text{is} \qquad J_\varphi(x) = \det\left(\frac{\partial \varphi_i}{\partial x_j}\right).$$

If φ is linear then there are $\phi_{ij} \in \mathbb{C}$ such that

$$\phi_i(x) = \sum_{j=1}^n \phi_{ij} x_j \qquad \text{and} \qquad J_\varphi = \det(\phi_{ij}).$$

The *chain rule* is the identity

$$J_{\theta \circ \varphi} = J_\theta(\varphi x) J_\varphi(x).$$

Let

$$\begin{array}{rccc} \theta\colon & V & \longrightarrow & V \\ & x & \longrightarrow & (e_1(x), \ldots, e_n(x)) \end{array} \qquad \text{and} \qquad \begin{array}{rccc} w\colon & V & \longrightarrow & V \\ & x & \longrightarrow & wx \end{array}$$

for $w \in W$. Then $\theta \circ w = \theta$ and so

$$J_\theta(x) = J_{\theta \circ w}(x) = J_\theta(wx) J_w(x) = J_\theta(wx)\det(w) = \det(w)(w^{-1}J_\theta)(x).$$

Thus J_θ is W-alternating and so J_θ is divisible by

$$\Delta = \prod_{\alpha \in R^+} \alpha^{r_\alpha - 1}. \qquad \text{Since} \quad \deg(J_\theta) = \sum_{i=1}^n (d_i - 1) = \operatorname{Card}(R^+),$$

and so $J_\theta = \lambda \cdot \Delta$ for some $\lambda \in \mathbb{C}$. ∎

Step 4. The polynomials $e_1, \ldots, e_n$ are algebraically independent if and only if $J_\theta \neq 0$.

Proof. $\Rightarrow$: Assume $e_1, \ldots, e_r$ are algebraically independent.
Then x_i are algebraic over $\mathbb{C}(e_1, \ldots, e_r)$.

$$\operatorname{trdeg}\left(\frac{\mathbb{C}(x_1, \ldots, x_n}{\mathbb{C}(e_1, \ldots, e_r)}\right) = \operatorname{trdeg}\left(\frac{\mathbb{C}(x_1, \ldots, x_n)}{\mathbb{C}}\right) - \operatorname{trdeg}\left(\frac{\mathbb{C}(e_1, \ldots, e_r)}{\mathbb{C}}\right) \geq n - r.$$

So $x_1, \dots, x_n$ are algebraic over $\mathbb{C}(e_1, \dots, e_r)$ if and only if $0 \geq n - r$, that is, if and only if $n = r$.

Let $m_i(t) \in S(V)^W[t]$ be the minimal polynomial of x_i over $\mathbb{C}(e_1, \dots, e_n)$, the field of fractions of $S(V)^W$. Then

$$\frac{\partial m_i}{\partial x_k} = \sum_{j=1}^{r} \frac{\partial m_i}{\partial e_j} \frac{\partial e_j}{\partial x_k} + \frac{\partial m_i}{\partial t} \frac{\partial t}{\partial x_k}$$

and

$$0 = \frac{\partial m_i(x_i)}{\partial x_k} = \sum_{j=1}^{r} \frac{\partial m_i}{\partial e_j}(x_i) \frac{\partial e_j}{\partial x_k} + m_i'(x_i)\delta_{ik}.$$

Thus

$$\det\left(\frac{\partial m_i}{\partial e_j}(x_i)\right) \cdot J_\theta = \det\big(- \mathrm{diag}(m_1'(x_1), \dots, m_n'(x_n)\big) = (-1)^n \prod_{i=1}^{r} m_i'(x_i).$$

Since $m_i(t)$ is the minimal polynomial of x_i, each factor $m_i'(x_i) \neq 0$ and, thus, $J_\theta \neq 0$.

$\Leftarrow$: Assume $e_1 \dots, e_n$ are algebraically dependent. Let $f(y_1, \dots, y_n)$ be of minimal degree such that $f(e_1, \dots, e_n) = 0$. Then

$$\frac{\partial f}{\partial y_i} \neq 0 \quad \text{for some } y_i, \qquad \text{and so} \qquad g_i = \frac{\partial f}{\partial y_i}(e_1, \dots, e_n) \neq 0 \quad \text{for some } i.$$

But

$$0 = \frac{\partial f(e_1, \dots, e_n)}{partial x_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(e_1, \dots, e_n) \frac{\partial e_i}{\partial x_j}, \qquad \text{and so} \qquad \sum_{i=1}^{n} g_i \frac{\partial e_i}{\partial x_j} = 0.$$

So g_i is a solution to the equation $(g_1, \dots, g_n)\big(\partial e_i / \partial x_j\big) = 0$ and so $J_\theta = 0$. ∎

∎

Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.