Notes: Toroidal Hecke algebras

Arun Ram*
Department of Mathematics
University of Wisconsin–Madison
Madison, WI 53706
ram@math.wisc.edu
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1. Grading toroidal Hecke algebras

The *toroidal Hecke algebra* ${}_XH_Y$ is the algebra given by generators

$$T_0, \ldots, T_n, \qquad X^\lambda, \quad \lambda \in P, \qquad \text{and} \qquad \Omega,$$

with relations

$$\begin{array}{ll}
(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{\frac{1}{2}}) = 0, & \text{for } 0 \le i \le n, \\
\underbrace{T_iT_jT_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\text{ factors}}, & \\
g_rT_ig_r^{-1} = T_j, & \text{if } g_r(\alpha_i) = \alpha_j, \\
T_iX^\lambda T_i = X^{\lambda-\alpha_i}, & \text{if } \langle \lambda, \alpha_i^\vee \rangle = 1, \\
T_iX^\lambda = X^\lambda T_i, & \text{if } \langle \lambda, \alpha_i^\vee \rangle = 0, \\
g_rX^\lambda g_r^{-1} = X_{g_r\lambda} = X^{u_r^{-1}\lambda}q^{\langle \omega_r, \lambda\rangle}, & \text{for } r \in P/Q,
\end{array}$$

Additional relations/definitions are

$$\langle \mu + j\delta, \lambda \rangle = \langle \mu, \lambda \rangle, \quad q_\alpha = q^{\langle \alpha, \alpha \rangle/2}, \quad t_\alpha = q_\alpha^{c_\alpha},$$

and

$$X^{\lambda+j\delta} = X^\lambda q^j, \quad \text{and} \qquad Y^{\lambda+j\delta} = Y^\mu q^{-j},$$

and

$$u_r = g_r^{-1}\omega_r, \qquad u_{r^*} = u_r^{-1}, \qquad \pi_{r^*} = \pi_r^{-1}, \qquad \text{for } r \in P/Q.$$

For $\lambda \in P$ define

$$Y^\lambda = \prod_{i=1}^n Y_i^{\langle \lambda, \alpha_i^\vee \rangle}, \quad \text{where} \quad Y_i = T_{\omega_i},$$

for $1 \le i \le n$. Then, for $\lambda \in P$ and $1 \le i \le n$,

$$\begin{array}{c}
T_i^{-1}Y^\lambda T_i^{-1} = Y^{\lambda-\alpha_i}, \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 1, \qquad \text{and} \\
T_iY^\lambda = Y^\lambda T_i, \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 0.
\end{array}$$

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Inside H let
$$\begin{aligned}
Y^{\alpha_0} &= q^{-1}Y^{-\phi}, \\
\Phi_i &= T_i + (t_i^{1/2} - t_i^{-1/2})\frac{1}{Y^{-\alpha_i}-1}, \qquad \text{for } 1 \le i \le n, \\
\Phi_0 &= X^{\phi}T_{s_\phi} - (t_0^{1/2} - t_0^{-1/2})\frac{1}{Y^{\alpha_0}-1}, \\
\gamma_i &= t_i^{1/2} + (t_i^{1/2} - t_i^{-1/2})\frac{1}{Y^{-\alpha_i}-1}, \qquad \text{for } 0 \le i \le n,
\end{aligned}$$
and let
$$s_i = \Phi_i\gamma_i^{-1}, \qquad \text{and} \qquad g_r = X^{\omega_r}T_{u_r^{-1}}.$$
Then, amazingly, the $s_0, s_1, \ldots, s_n$ and the g_r generate a copy of $\tilde{W}$ inside ${}_XH_Y$.

The graded toroidal Hecke algebra ${}_XH_y$

The graded toroidal Hecke algebra ${}_XH_y$ is the algebra generated by
$$\mathbb{C}[\tilde{W}] \qquad \text{and} \qquad y_1, \ldots, y_n$$
with relations
$$\begin{aligned}
s_jy_\lambda - y_{s_j\lambda}s_j &= -c_{\alpha_j}\langle\lambda, \alpha_j^\vee\rangle, \qquad \text{for } 0 \le j \le n, \\
g_ry_\lambda &= y_{g_r\lambda}g_r, \qquad \text{for } r \in P/Q.
\end{aligned}$$
where $\langle\lambda, \alpha_0^\vee\rangle = -\langle\lambda, \phi^\vee\rangle$, for $\lambda \in P$. If we set
$$q = e^{\xi}, \qquad Y^\lambda = e^{-\xi y_\lambda},$$
then ${}_XH_y$ is the $\xi \to 0$ limit of ${}_XH_Y$ and
$$y_{\lambda+j\delta} = j + \sum_{i=1}^{n}\langle\lambda, \alpha_i^\vee\rangle y_i,$$

The graded graded toroidal Hecke algebra ${}_xH_y$

The *graded graded toroidal Hecke algebra* is the algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$, and W, with relations
$$wx = w(x)w, \quad wy = w(y)w, \quad x_1x_2 = x_2x_1, \quad y_1y_2 = y_2y_1,$$
$$yx - xy = \langle y, x, \rangle - \sum_{\alpha\in R^+} c_\alpha\langle y, \alpha\rangle\langle\alpha^\vee, x\rangle s_\alpha,$$
for $x_1, x_2, x \in \mathfrak{h}^*$, $y_1, y_2, y \in \mathfrak{h}$ and $w \in W$. If
$$X^\lambda = e^{sx_\lambda} \qquad \text{then} \qquad {}_xH_y = \lim_{s\to 0} {}_XH_y.$$

If we set
$$q = e^{h}, \qquad Y^\lambda = e^{-\sqrt{h}y_\lambda}, \qquad \text{and} \qquad X^\lambda = e^{\sqrt{h}x_\lambda},$$

then
$$ {}_xH_y = \lim_{h\to 0} {}_XH_Y. $$

The polynomial representation

Let
$$ \alpha_0 = -\phi + \delta \qquad \text{and} \qquad \rho_c = \frac{1}{2}\sum_{\alpha\in R^+} c_\alpha \alpha. $$

Let
$$ H_Y \quad \text{be the subalgebra of } H \text{ generated by } T_i,\ 1\le i\le n, \text{ and } Y^\lambda,\ \lambda\in P. $$

There is a one dimensional H_Y module $\mathbf{1}$ given by
$$ \begin{array}{rccc} \mathbf{1}\colon & H_Y & \longrightarrow & \mathbb{C} \\ & T_i & \longmapsto & t_i^{1/2} \\ & Y^{\omega_i} & \longmapsto & t_i^{1/2}????, \end{array} $$

Then the induced module ${}_XH_Y \otimes_{H_Y} \mathbf{1}$ is the vector space $\mathbb{C}[X] = \mathbb{C}\text{-span}\{X^\lambda \mid \lambda\in P\}$ with H-action given by
$$ \begin{aligned} T_i &= t_i^{1/2}s_i + (t_i^{1/2} - t_i^{-1/2})\frac{1}{X^{\alpha_i}-1}(s_i - 1), \qquad \text{for } 0\le i\le n, \\ T_0 &= t_0^{1/2}s_0 + (t_0^{1/2} - t_0^{-1/2})\frac{1}{qX^{-\phi}-1}(s_0 - 1), \end{aligned} $$

where
$$ s_0(X^\lambda) = X^{\lambda - \langle\lambda,\phi^\vee\rangle\phi} q^{\langle\lambda,\phi\rangle}, $$

for $\lambda\in P$. The *trigonometric difference Dunkl operators* are the
$$ \begin{array}{rccc} \Theta^\lambda\colon & \mathbb{C}[X] & \longrightarrow & \mathbb{C}[X] \\ & f & \longrightarrow & Y^\lambda f, \end{array} \qquad \text{for } \lambda\in P. $$

Define operators $\partial_\lambda\colon \mathbb{C}[X]\to\mathbb{C}[X]$, $\lambda\in P$, by
$$ \partial_\lambda(X^\mu) = \langle\lambda,\mu\rangle X^\mu, \qquad \text{for } \mu\in P. $$

Then $w(\partial_\lambda) = \partial_{w\lambda}$ for $\lambda\in P$ and $w\in W$ and the operator $\mathcal{D}_\lambda\colon \mathbb{C}[X]\to\mathbb{C}[X]$ given by
$$ \mathcal{D}_\lambda = \lim_{h\to 0}\Theta^\lambda = \partial_\lambda + \sum_{\alpha\in R^+} \frac{c_\alpha\langle\lambda,\alpha^\vee}{1-X^{-\alpha}}(1-s_\alpha) - \langle\lambda,\rho_c\rangle $$

is the limit of $\Theta^\lambda\colon \mathbb{C}[X]\to\mathbb{C}[X]$ at $h\to 0$.

Define operators $d_\mu\colon \mathbb{C}[x]\to\mathbb{C}[x]$ by
$$ d_\mu(x_\lambda) = \langle\mu,\lambda\rangle x_\lambda, \qquad \text{for } \lambda,\mu\in\mathfrak{h}^*, $$

so that
$$ d_\mu = \lim_{s\to 0} s\partial_\mu \qquad \text{under the substitution } X^\lambda = e^{sx_\lambda}. $$

Then

$$D_\lambda = \lim_{s\to 0} s\mathcal{D}_\lambda = d_\lambda + \sum_{\alpha\in R^+} \frac{c_\alpha\langle\lambda,\alpha^\vee\rangle}{x_\alpha}(1-s_\alpha) \tag{1.1}$$

and

$$\mathcal{D}_\lambda = \frac{1}{s}D_\lambda - \langle\rho_c,\lambda\rangle + \sum_{\alpha\in R^+} c_\alpha\langle\lambda,\alpha\rangle \sum_{m\in\mathbb{Z}_{\ge 0}} \frac{B_m}{m!}(-sx_\alpha)^m(1-s_\alpha), \tag{1.2}$$

where B_m are the Bernoulli numbers defined by

$$\frac{x}{e^x-1} = \sum_{m\in\mathbb{Z}_{\ge 0}} \frac{B_m}{m!}.$$

2. Graded toroidal Hecke algebras

Let W be a real reflection group with reflection representation $\mathfrak{h}$. Let R^+ be the set of reflections in W. Let $\hat{W}$ be a index set for the irreducible W-modules and let W^λ denoe the simple W-module indexed by $\lambda\in\hat{W}$. Fix a W-invariant function

$$\begin{array}{rcl} c\colon R^+ & \longrightarrow & \mathbb{C} \\ \alpha & \longmapsto & c_\alpha \end{array}$$

The *Casimir element* of $\mathbb{C}W$ is

$$\kappa_c = \sum_\alpha c_\alpha(1-s_\alpha) \quad \in Z(\mathbb{C}W).$$

For each $\lambda\in\hat{W}$ let $\kappa_c(\lambda)$ be the constant such that

$$\kappa_c \text{ acts as } \kappa_c(\lambda)\cdot \mathrm{Id} \text{ on } W^\lambda.$$

The *toroidal Hecke algebra* $\mathbb{H}_c$ is the algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$, and W, with relations

$$wx = w(x)w, \quad wy = w(y)w, \quad x_1x_2 = x_2x_1, \quad y_1y_2 = y_2y_1,$$

$$yx - xy = \langle y, x, \rangle - \sum_{\alpha\in R^+} c_\alpha\langle y,\alpha\rangle\langle\alpha^\vee, x\rangle s_\alpha,$$

for $x_1, x_2, x\in\mathfrak{h}^*$, $y_1, y_2, y\in\mathfrak{h}$ and $w\in W$. The "PBW"-theorem for $\mathbb{H}_c$ is that, as vector spaces,

$$\mathbb{H}_c = \mathbb{C}[\mathfrak{h}]\otimes\mathbb{C}W\otimes\mathbb{C}[\mathfrak{h}^*], \qquad \text{where} \qquad \begin{array}{l} \mathbb{C}[\mathfrak{h}] = (\text{the subalgebra generated by } x\in\mathfrak{h}^*) \quad \text{and} \\ \mathbb{C}[\mathfrak{h}^*] = (\text{the subalgebra generated by } y\in\mathfrak{h}). \end{array}$$

The principal $\mathfrak{sl}_2$

Let

$\{x_i\}$ be a basis of $\mathfrak{h}^*$,
$\{y_i\}$ be the basis of $\mathfrak{h}$ dual to $\{x_i\}$,
$\{x_i^*\}$ be the basis of $\mathfrak{h}^*$ dual to $\{x_i\}$ with respect to $\langle,\rangle$,
$\{x_i\}$ be the basis of $\mathfrak{h}$ dual to $\{y_i\}$ with respect to $\langle,\rangle$,

Then the elements

$$e = \sum_i x_i x_i^*, \qquad f = \sum_i y_i y_i^*, \qquad h = \frac{1}{2}\sum_i x_i y_i + y_i x_i,$$

form an $\mathfrak{sl}_2$-triple in $\mathbb{H}_c$.

The spherical Hecke algebra

Let

$$\mathbf{1} = \frac{1}{|W|}\sum_{w\in W} w \qquad \text{and} \qquad \epsilon = \frac{1}{|W|}\sum_{w\in W} \det(w)w,$$

be the minimal idempotents in $\mathbb{C}W$ corresponding to the trivial and the det representations, respectively.

The *spherical Hecke algebra* is $\mathbf{1}\mathbb{H}_c\mathbf{1}$ and the det-*spherical Hecke algebra* is $\epsilon\mathbb{H}_c\epsilon$. Since $\mathbb{H}_c\epsilon$ is an $(\mathbb{H}_c, \epsilon\mathbb{H}_c\epsilon)$ there are functors

$$\begin{array}{ccc} \mathbb{H}_c\text{-mod} & \longrightarrow & \epsilon\mathbb{H}_c\epsilon\text{-mod} \\ M & \longmapsto & \epsilon M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \epsilon\mathbb{H}_c\epsilon\text{-mod} & \longrightarrow & \mathbb{H}_c\text{-mod} \\ N & \longmapsto & \mathbb{H}_c\epsilon \otimes_{\epsilon\mathbb{H}_c\epsilon} N \end{array}$$

Let

$$\theta_c \colon \mathbf{1}\mathbb{H}_{c-1}\mathbf{1} \longrightarrow \operatorname{End}(\mathbb{C}[\mathfrak{h}]) \qquad \text{and} \qquad \Theta_c^- \colon \epsilon\mathbb{H}_c\epsilon \longrightarrow \operatorname{End}(\mathbb{C}[\mathfrak{h}])$$

be the *spherical* and the *antispherical* Harish-Chandra isomorphisms, respectively. (Should the constant function 1 be called ρ here???? or $\rho^\vee$???)

Theorem 2.1. (Shift isomorphism) *The map*

$$\Theta_c^{-1} \circ \Theta_c^- \colon \epsilon\mathbb{H}_c\epsilon \xrightarrow{\sim} \mathbf{1}\mathbb{H}_{c-1}\mathbf{1}$$

is a well defined algebra isomorphism.

The category $\mathcal{O}_c$

The category $\mathcal{O}_c$ is the category of $\mathbb{H}_c$-modules M such that

(a) For each $m \in M$, $\mathbb{C}[\mathfrak{h}^*]m$ is finite dimensional,

(b) If $p \in \mathbb{C}[fh^*]^W$ then $p - p(0)$ acts locally nilpotently on M (What??? does this mean???).

Let $\hat{W}$ be a index set for the irreducible W-modules and let W^λ denoe the simple W-module indexed by $\lambda \in \hat{W}$. The *standard modules* are

$$M_c(\lambda) = \mathbb{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\otimes\mathbb{C}W} W^\lambda, \qquad \text{for } \lambda \in \hat{W},$$

where teh $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W$ action on W^λ is given by

$$pw \cdot m = p(0)wm, \qquad \text{for } p \in \mathbb{C}[\mathfrak{h}^*],\ w \in W,\ m \in W^\lambda.$$

Theorem 2.2. (Berest-Etingof-Ginzburg)

(a) $M_c(\lambda)$ has a unique simple quotient $L_c(\lambda)$.

(b) $\{L_c(\lambda) \mid \lambda \in \hat{W}\}$ *are the simple objects in* $\mathcal{O}_c$.
(c) Every object in $\mathcal{O}_c$ *has finite length.*

2. Gordon's results

By an old theorem of Steinberg?? (exercise in Bourbaki) the $\wedge^i\mathfrak{h}$, $0 \le i \le n$ are nonisomorphic irreducible representations of W. As a $\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W$-module

$$M_c(\wedge^i\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}] \otimes \wedge^i\mathfrak{h}.$$

The graded W-character of $M_c(\wedge^i\mathfrak{h})$ (with respect to which grading??? there are two that coming from h-eigenspaces and the other coming from $\mathbb{H}_c$ as a deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$, how do these two gradings relate, I'm totally confused???) is

$$\mathrm{ch}(M_c(\wedge^i\mathfrak{h}); w, t) = \frac{t^{\kappa_c(\wedge^i\mathfrak{h})}\mathrm{ch}(\wedge^i\mathfrak{h}, t)}{\det(1-tw)}. \tag{2.1}$$

Let $m \in \mathbb{Z}_{\ge 0}$ and

$$\text{assume} \quad c = \frac{1+mh}{h}, \qquad \text{where} \quad h = \frac{2\mathrm{Card}(R^+)}{\dim(\mathfrak{h})},$$

is the Coxeter number. Then
(a) If $\lambda \ne \mu$ and there is a nonzero $\mathbb{H}_c$-module homomorphism

$$M_c(\lambda) \to M_c(\mu) \quad \text{then} \quad \lambda = \wedge^i\mathfrak{h} \quad \text{and} \quad \mu = \wedge^j\mathfrak{h},$$

for some i and j.
(b) If $M_c(\lambda)$ is not simple then $W^\lambda \cong \wedge^i\mathfrak{h}$ for some $0 \le i \le n$,
(c) If $L_c(\lambda)$ is finite dimensional then $W^\lambda \cong \wedge^i\mathfrak{h}$ for some $0 \le i \le n$,
(d) $[M_c(\wedge^i\mathfrak{h}) : L_c(\wedge^j\mathfrak{h})] = [\epsilon M_c(\wedge^i\mathfrak{h}) : \epsilon L_c(\wedge^j\mathfrak{h})]$,
(e) the Hilbert series of $M_c(\wedge^i\mathfrak{h})$ with respect to the h-eigenspaces is

$$P(\epsilon M_c(\wedge^i\mathfrak{h}), t) = \frac{t^{-m|R^+|+i(mh+1)}}{\prod_{j=1}^n (1-t^{d_i})} e_{n-i}(t^{e_1}, \ldots, t^{e_n}),$$

where $d_1, \ldots, d_n$ are the degrees of W, $e_i = d_i - 1$ are the exponents of W and $e_r(x_1, \ldots, x_n)$ is the rth elementary symmetric function.
(f) $\epsilon L_c(\wedge^i\mathfrak{h}) \ne 0$,

Now

$$\text{assume} \quad c = \frac{1+h}{h}.$$

Then
(a) there is a 1-dimensional $\epsilon\mathbb{H}_c\epsilon$-module ϵ_1,
(b) $L_c(\wedge^0\mathfrak{h}) = \mathbb{H}_c\epsilon \otimes_{\epsilon\mathbb{H}_c\epsilon} \epsilon_1$,

(c) $L_c(\wedge^9\mathfrak{h}) = \sum_{i=0}^{n}(-1)^i M_c(\wedge^i\mathfrak{h})$.

(d) The Hilbert series of $L_c(\wedge^0\mathfrak{h})$ is

$$P(L_c(\wedge^0\mathfrak{h};t) = t^{-|R^+|}(1+t+t^2+\cdots+t^h)^n.$$

(e) The graded character of $L_c(\wedge^0\mathfrak{h})$ is

$$\mathrm{ch}(L_c(\wedge^0\mathfrak{h};w,t) = t^{-|R^+|}\frac{\det(1-t^{h+1}w)}{\det(1-tw)},$$

and, by putting $t=1$, the W-character of $L_c(\wedge^0\mathfrak{h})$ is

$$\mathrm{ch}(L_c(\wedge^0\mathfrak{h};w) = h^{\dim\ker(1-w)},$$

and thus, by comparison with Sommers [ref??],

$$L_c(\wedge^0\mathfrak{h}) \cong \frac{Q}{(h+1)Q}, \qquad \text{as } W\text{-modules},$$

when W is a Weyl group and Q is the root lattice of W.

The connection to diagonal harmonics

The filtration on $\mathbb{H}_c$ given by

$$\deg(x)=1, \qquad \deg(y)=1, \quad \deg(w)=0, \qquad x\in\mathfrak{h}^*,\ y\in\mathfrak{h},\ w\in W,$$

has associated graded

$$\mathrm{gr}(\mathbb{H}_c) = \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]*W,$$

the "semidirect" product of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ and $\mathbb{C}W$. Let

$$c=\frac{1+h}{h} \qquad \text{and} \qquad L=L_c(\wedge^0\mathfrak{h}) = \mathbb{H}_c\epsilon\otimes_{\epsilon\mathbb{H}_c\epsilon}\epsilon_1.$$

Then the surjection

$$(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]*W)\epsilon\otimes_{\epsilon(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]*W)\epsilon}\epsilon_1 \longrightarrow \mathrm{gr}(L)$$

and the graded $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]*W$ isomorphisms

$$\begin{array}{rcll} \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\otimes\epsilon & \xrightarrow{\sim} & (\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]*W)\epsilon = \mathrm{gr}(\mathbb{H}_c\epsilon), & \qquad\text{and} \\ \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W\epsilon & \xrightarrow{\sim} & \epsilon(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]*W)\epsilon = \mathrm{gr}(\epsilon\mathbb{H}_c\epsilon), & \end{array}$$

provide a graded surjection

$$\frac{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]}{\langle\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]_+^W\rangle} \cong \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\otimes_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W}\epsilon_1 \longrightarrow \mathrm{gr}(L)\otimes\epsilon.$$

The KZ-connection

There is an injective algebra homomorphism

$$\mathbb{H}_c \longrightarrow D(\mathfrak{h}^{\mathrm{reg}}) \otimes \mathbb{C}W$$

and a corresponding *localization functor*

$$\begin{array}{ccc} \mathbb{H}_c\text{-mod} & \longrightarrow & \{W\text{-equivariant } D\text{-modules on } \mathfrak{h}^{\mathrm{reg}}\} \\ M & \longmapsto & M\big|_{\mathfrak{h}^{\mathrm{reg}}} \end{array}$$

The *KZ-connection with values in* W^λ is the flat connection on

$$M_c(\lambda)\big|_{\mathfrak{h}^{\mathrm{reg}}} = (\text{trivial vector bundle } \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \otimes W^\lambda).$$

The corresponding monodromy representation (in a fiber over a point in $\mathfrak{h}^{\mathrm{reg}}/W$), $Mon_c(\lambda)$, of the braid group, $\pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$, factors through the Hecke algebra $H_W(e^{2\pi i c})$.

6. References

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