Taylor's theorem and the limit formula

Arun Ram
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

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The derivative of f with respect to x is $\frac{df}{dx}$. It is common to write $f'(x)$ in place of $\frac{df}{dx}$.

$$f'(x) = \frac{df}{dx}.$$

The *second derivative of f with respect to x* is

$$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right),$$

the derivative of the derivative of f. Both $\frac{d^2f}{dx^2}$ and $f''(x)$ are notations for the same thing, the second derivative of f.

The *third derivative of f with respect to x* is

$$f'''(x) = \frac{d^3f}{dx^3} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right),$$

the derivative of the second derivative of f. Use the notations $\frac{d^3f}{dx^3}$ and $f'''(x)$ interchangably for the third derivative of f.

The *fourth derivative of f with respect to x* is

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = \frac{d}{dx}\left(\frac{d^3f}{dx^3}\right),$$

the derivative of the third derivative of f.

Let a be a number. Then f *evaluated at a* is

$$f(a) = f\big|_{x=a} = c_0 + c_1a + c_2a^2 + c_3x^3 + \cdots,$$

if $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. Use both notations, $f(a)$ and $f\big|_{x=a}$, interchangably, for f evaluated at a.

Example: If $f(x) = 7x^3 + 3x^2 + 5x + 12$ and $a = 3$ then

$$\begin{aligned} f(3) &= 7\cdot 3^3 + 3\cdot 3^2 + 5\cdot 3 + 12 = 8\cdot 3^3 + 27 = 9\cdot 3^3 = 3^5, \\ f\Big|_{x=3} &= 7\cdot 3^3 + 3\cdot 3^2 + 5\cdot 3 + 12 = 8\cdot 3^3 + 27 = 9\cdot 3^3 = 3^5. \end{aligned}$$

$$\begin{aligned} \frac{df}{dx} &= 21x^2 + 6x + 5, & \frac{df}{dx}\Big|_{x=3} &= 21\cdot 3^2 + 6\cdot 3 + 5 = 189 + 23 = 202, \\ f' &= 21x^2 + 6x + 5, & f'(3) &= 21\cdot 3^2 + 6\cdot 3 + 5 = 189 + 23 = 202, \end{aligned}$$

$$\begin{aligned} \frac{d^2f}{dx^2} &= 42x + 6, & \frac{d^2f}{dx^2}\Big|_{x=3} &= 42\cdot 3 + 6 = 132, \\ f'' &= 42x + 6, & f''(3) &= 42\cdot 3 + 6 = 132, \end{aligned}$$

$$\begin{aligned} \frac{d^3f}{dx^3} &= 42, & \frac{d^3f}{dx^3}\Big|_{x=3} &= 42, \\ f''' &= 42, & f'''(3) &= 42, \end{aligned}$$

$$\begin{aligned} \frac{d^4f}{dx^4} &= 0, & \frac{d^4f}{dx^4}\Big|_{x=3} &= 0, \\ f^{(4)} &= 0, & f^{(4)}(3) &= 0. \end{aligned}$$

Taylor's and Macluarin's theorems and the limit formula for the derivative

If $\quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \cdots$

then

$$f(a) = c_0,$$

$$\frac{df}{dx}\Big|_{x=a} = \big(c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots\big)\Big|_{x=a} = c_1,$$

$$\frac{d^2f}{dx^2}\Big|_{x=a} = \big(2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \cdots\big)\Big|_{x=a} = 2c_2,$$

$$\frac{d^3f}{dx^3}\Big|_{x=a} = \big(3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + 6\cdot 5\cdot 4c_6(x-a)^3 + \cdots\big)\Big|_{x=a} = 3\cdot 2c_3,$$

$$\frac{d^4f}{dx^4}\Big|_{x=a} = \big(4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5(x-a) + 6\cdot 5\cdot 4\cdot 3c_4(x-a)^2 + \cdots\big)\Big|_{x=a} = 4\cdot 3\cdot 2c_4,$$

and we can continue this process to find

$$\left.\frac{d^k f}{dx^k}\right|_{x=a} = k!c_k, \qquad \text{for } k = 1, 2, 3, \ldots.$$

Dividing both sides by $k!$ gives

$$c_k = \frac{1}{k!}\left(\left.\frac{d^k f}{dx^k}\right|_{x=a}\right).$$

So

$$f(x) = f(a) + \left(\left.\frac{df}{dx}\right|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\left.\frac{d^2 f}{dx^2}\right|_{x=a}\right)(x-a)^2 + \frac{1}{3!}\left(\left.\frac{d^3 f}{dx^3}\right|_{x=a}\right)(x-a)^3 + \cdots,$$

or, equivalently,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + \cdots.$$

Now subtract $f(a)$ from both sides:

$$f(x) - f(a) = f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + \cdots.$$

Divide both sides by $x - a$.

$$\frac{f(x) - f(a)}{x - a} = f'(a) + \frac{1}{2!}f''(a)(x-a) + \frac{1}{3!}f'''(a)(x-a)^2 + \frac{1}{4!}f^{(4)}(a)(x-a)^3 + \cdots.$$

Evaluate both sides at $x = a$.

$$\left.\frac{f(x) - f(a)}{x - a}\right|_{x=a} = f'(a) + 0 + 0 + 0 + 0 + \cdots.$$

So $\qquad f'(a) = \left.\dfrac{f(x) - f(a)}{x - a}\right|_{x=a}.$

Let $x = a + h$. Then $\qquad f'(a) = \left.\dfrac{f(a+h) - f(a)}{a + h - a}\right|_{a+h=a}.$

So $\qquad \left.\dfrac{df}{dx}\right|_{x=a} = \left.\dfrac{f(a+h) - f(a)}{h}\right|_{h=0}.$

Another way to write this is

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}.$$

Example: Suppose you want to know what f I'm thinking of and I refuse to tell you.

You ask me what $f(0)$ is and I say "6".
You ask me what $f'(0)$ is and I say "10".
You ask me what $f''(0)$ is and I say "31".
You ask me what $f'''(0)$ is and I say "5".
You ask me what $f^{(4)}(0)$ is and I say "7".
You ask me what $f^{(5)}(0)$ is and I say "0".
You ask me what $f^{(6)}(0)$ is and I say "0".
You ask me what $f^{(7)}(0)$ is and I say "0, they are all coming out to 0 now.".

At this point you win because you know that

$$\begin{aligned}
f(x) &= f(0) + f'(0)(x-0) + \frac{1}{2!}f''(0)(x-0)^2 + \frac{1}{3!}f'''(0)(x-0)^3 + \cdots \\
&= 6 + 10(x-0) + \frac{1}{2!}31(x-0)^2 + \frac{1}{3!}5(x-0)^3 + \frac{1}{4!}7(x-0)^4 \\
&\qquad + \frac{1}{5!}\cdot 0(x-0)^5 + \frac{1}{6!}\cdot 0(x-0)^6 + \frac{1}{7!}\cdot 0(x-0)^7 + 0 + 0 + \cdots \\
&= 6 + 10x + \frac{31}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{24}x^4,
\end{aligned}$$

and so you have found out what f is.

Example: Suppose you want to know what f I'm thinking of and I refuse to tell you.

You ask me what $f(0)$ is and I say "I won't tell you, but $f(3) = 4$".
You ask me what $f'(0)$ is and I say "I won't tell you, but $\left.\frac{df}{dx}\right|_{x=3} = 2$".
You ask me what $f''(0)$ is and I say "I won't tell you, but $\left.\frac{d^2f}{dx^2}\right|_{x=3} = 5$".
You ask me what $f'''(0)$ is and I say

"I won't tell you, but $\left.\frac{d^3f}{dx^3}\right|_{x=3} = 0$ and all the rest of the $\left.\frac{d^kf}{dx^k}\right|_{x=3}$ are coming out to 0".

At this point you win because you know that

$$\begin{aligned}
f(x) &= f\big|_{x=3} + \left(\left.\frac{df}{dx}\right|_{x=3}\right)(x-3) + \frac{1}{2!}\left(\left.\frac{d^2f}{dx^2}\right|_{x=3}\right)(x-3)^2 + \frac{1}{3!}\left(\left.\frac{d^3f}{dx^3}\right|_{x=3}\right)(x-3)^3 + \cdots \\
&= 2 + 5(x-3) + \frac{1}{2!}5(x-3)^2 + \frac{1}{3!}\cdot 0(x-3)^3 + 0 + 0 + \cdots \\
&= 2 + 5x - 15 + \frac{5}{2}(x^2 - 6x + 9) + 0 + 0 + \cdots \\
&= -13 + 5x + \frac{5}{2}x^2 - 15x + \frac{45}{2} \\
&= \frac{5}{2}x^2 - 10x + \frac{19}{2},
\end{aligned}$$

and so you know what f is.