Symmetric functions
Lecture Notes

Arun Ram*
Department of Mathematics
University of Wisconsin–Madison
Madison, WI 53706
ram@math.wisc.edu
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## 1. Symmetric functions

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the $\mathbb{Z}$-basis of $\mathbb{Z}^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z}\}$ given by $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the $i$th entry, so that

$$\begin{aligned} \mathbb{Z}^n &= \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\}, \\ \text{and let} \quad P^+ &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}, \\ \text{and} \quad P^{++} &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{Z}^n \mid \lambda_1 > \cdots > \lambda_n\}. \end{aligned} \tag{1.1}$$

Then $P^+$ is a set of representatives of the orbits of the action of the symmetric group $S_n$ on $\mathbb{Z}^n$ given by permuting the coordinates,

$$w\varepsilon_i = \varepsilon_{w(i)}, \qquad \text{for } w \in S_n,\ 1 \leq i \leq n. \tag{1.2}$$

There is a bijection

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{where} \quad \rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}. \tag{1.3}$$

Let

$$\mathbb{Z}[X] = \mathbb{Z}\text{-span}\{x^\lambda \mid \lambda \in \mathbb{Z}^n\} \qquad \text{with} \quad x^\lambda x^\mu = x^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in \mathbb{Z}^n. \tag{1.4}$$

For $1 \leq i \leq n$ write

$$x_i = x^{\varepsilon_i} \qquad \text{so that} \qquad x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \quad \text{for } \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n,$$

and $\mathbb{Z}[X] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The action of $S_n$ on $\mathbb{Z}^n$ induces an action of $S_n$ on $\mathbb{Z}[X]$ given by

$$wx^\lambda = x^{w\lambda}, \qquad \text{for } w \in S_n,\ \lambda \in \mathbb{Z}^n. \tag{1.5}$$

The ring of *symmetric functions* is

$$\mathbb{Z}[X]^{S_n} = \{f \in \mathbb{Z}[X] \mid wf = f \text{ for all } w \in S_n\}, \tag{1.6}$$

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Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in S_n\lambda} x^\gamma, \qquad \text{for } \lambda \in P^+,$$

where $S_n\lambda$ is the orbit of $\lambda$ under the action of $S_n$. Then

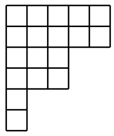
$$\{m_\lambda \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[X]^{S_n}. \tag{1.7}$$

*Partitions*

A *partition* is a collection $\mu$ of boxes in a corner where the convention is that gravity goes up and to the left. As for matrices, the rows and columns of $\mu$ are indexed from top to bottom and left to right, respectively.

$$\begin{array}{ll} \text{The } \textit{parts} \text{ of } \mu \text{ are} & \mu_i = (\text{the number of boxes in row } i \text{ of } \mu), \\ \text{the } \textit{length} \text{ of } \mu \text{ is} & \ell(\mu) = (\text{the number of rows of } \mu), \\ \text{the } \textit{size} \text{ of } \mu \text{ is} & |\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)} = (\text{the number of boxes of } \mu). \end{array} \tag{1.8}$$

Then $\mu$ is determined by (and identified with) the sequence $\mu = (\mu_1, \ldots, \mu_\ell)$ of positive integers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$, where $\ell = \ell(\mu)$. For example,

$(5, 5, 3, 3, 1, 1) =$  .

A *partition of* $k$ is a partition $\lambda$ with $k$ boxes. Make the convention that $\lambda_i = 0$ if $i > \ell(\lambda)$. The *dominance order* is the partial order on the set of partitions of $k$,

$$P^+(k) = \{\text{partitions of } k\} = \{\lambda = (\lambda_1, \ldots, \lambda_\ell) \mid \lambda_1 \geq \cdots \geq \lambda_\ell > 0,\ \lambda_1 + \ldots + \lambda_\ell = k\},$$

given by

$$\lambda \geq \mu \qquad \text{if} \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for all } 1 \leq i \leq \max\{\ell(\lambda), \ell(\mu)\}.$$

PUT THE PICTURE OF THE HASSE DIAGRAM FOR $k = 6$ HERE.

*Tableaux*

Let $\lambda$ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ be a sequence of nonnegative integers. A *column strict tableau of shape* $\lambda$ *and weight* $\mu$ is a filling of the boxes of $\lambda$ with $\mu_1$ 1s, $\mu_2$ 2s, ..., $\mu_n$ $n$s, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If $T$ is a column strict tableau write $\mathrm{shp}(T)$ and $\mathrm{wt}(T)$ for the shape and the weight of $T$ so that

$$\begin{aligned} \mathrm{shp}(T) &= (\lambda_1, \ldots, \lambda_n), && \text{where} \quad \lambda_i = \text{number of boxes in row } i \text{ of } T, \quad \text{and} \\ \mathrm{wt}(T) &= (\mu_1, \ldots, \mu_n), && \text{where} \quad \mu_i = \text{number of } i\text{s in } T. \end{aligned}$$

For example,

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & 3 & 3 & 4 \\ \cline{1-7} 3 & 3 & 3 & 4 & 4 & 4 & 5 \\ \cline{1-7} 4 & 5 & 5 & 6 \\ \cline{1-4} 6 & 7 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \text{has} \quad \begin{array}{l} \mathrm{shp}(T) = (9,7,7,4,2,1,0) \quad \text{and} \\ \mathrm{wt}(T) = (7,6,5,5,3,2,2). \end{array}$$

For a partition $\lambda$ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } T \mid \mathrm{shp}(T) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } T \mid \mathrm{shp}(T) = \lambda \text{ and } \mathrm{wt}(T) = \mu\}, \end{aligned} \tag{1.9}$$

## 2. Symmetric functions: Take 2

A *lattice* is a free $\mathbb{Z}$-module. Let $P$ be a lattice with a ($\mathbb{Z}$-linear) action of a finite group $W$ so that $P$ is a module for the group algebra $\mathbb{Z}W$. Extending coefficients, define

$$\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} P \qquad \text{and} \qquad \mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*,$$

so that $\mathfrak{h}_{\mathbb{R}}^*$ and $\mathfrak{h}^*$ are vector spaces which are modules for the group algebras $\mathbb{R}W$ and $\mathbb{C}W$, respectively.

Assume that the action of $W$ on $\mathfrak{h}_{\mathbb{R}}^*$ has fundamental regions???, and fix a fundamental region $C$ in $\mathfrak{h}_{\mathbb{R}}^*$. Define

$$P^+ = P \cap \overline{C} \qquad \text{and} \qquad P^{++} = P \cap C$$

so that $P^+$ is a set of representatives of the orbits of the action of $W$ on $P$. Assume???? that $P^+$ is a cone in $P$ (a module for the monoid $\mathbb{Z}_{\geq 0}$). A set of *fundamental weights* is a set of $\omega_1, \ldots, \omega_n$ generators of (the $\mathbb{Z}_{\geq 0}$-module) $P^+$ which also form a $\mathbb{Z}$-basis of $P$. There is a bijection

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{where} \quad \rho = \omega_1 + \ldots + \omega_n. \tag{2.1}$$

Let $\langle , \rangle \colon \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ be a $W$-invariant symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$ (such that the restriction to $P$ is a perfect pairing??? with values in $\mathbb{Z}$???). The *simple coroots* are $\alpha_1^\vee, \ldots, \alpha_n^\vee$ the dual basis to the fundamental weights,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{2.2}$$

Define

$$\overline{C^\vee} = \sum_{i=1}^n \mathbb{R}_{\leq 0} \alpha_i^\vee \qquad \text{and} \qquad C^\vee = \sum_{i=1}^n \mathbb{R}_{<0} \alpha_i^\vee. \tag{2.3}$$

The *dominance order* is the partial order on $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\lambda \geq \mu \qquad \text{if} \qquad \mu \in \lambda + \overline{C^\vee}. \tag{2.4}$$

The group algebra of the abelian group $P$ is

$$\mathbb{Z}[P] = \mathbb{Z}\text{-span}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in P. \tag{2.5}$$

The action of $W$ on $P$ induces an action of $W$ on $\mathbb{Z}[P]$ given by

$$wX^\lambda = X^{w\lambda}, \qquad \text{for } w \in W,\ \lambda \in P. \tag{2.6}$$

The ring of *symmetric functions* is

$$\mathbb{Z}[P]^W = \{f \in \mathbb{Z}[P] \mid wf = f \text{ for all } w \in W\}, \tag{2.7}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in W\lambda} X^\gamma, \qquad \text{for } \lambda \in P^+,$$

where $W\lambda$ is the orbit of $\lambda$ under the action of $W$. Then

$$\{m_\lambda \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[P]^W. \tag{2.8}$$

## 3. Type $Sp_{2n}(\mathbb{C})$

Let $W = WC_n$ be the group of $n \times n$ matrices with

(a) exactly one nonzero entry in each row and each column,

(b) the nonzero entries are $\pm 1$.

Then $W = WC_n = O_n(\mathbb{Z})$, the group of orthogonal matrices with entries in $\mathbb{Z}$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the $\mathbb{Z}$-basis of $\mathbb{Z}^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z}\}$ given by $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the $i$th entry, so that

$$\begin{aligned} P &= \mathbb{Z}^n = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\}, \\ \text{and let} \quad P^+ &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}, \\ \text{and} \quad P^{++} &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{Z}^n \mid \lambda_1 > \cdots > \lambda_n > 0\}. \end{aligned} \tag{3.1}$$

Then $P^+$ is a set of representatives of the orbits of the action of the natural action of $W$ on $P$. There is a bijection

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{where} \quad \rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n. \tag{3.2}$$

Let

$$\mathbb{Z}[P] = \mathbb{Z}\text{-span}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in P. \tag{3.3}$$

For $1 \leq i \leq n$ write

$$x_i = X^{\varepsilon_i} \qquad \text{so that} \qquad X^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \quad \text{for } \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n,$$

and $\mathbb{Z}[P] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The action of $W$ on $P$ induces an action of $W$ on $\mathbb{Z}[P]$ given by

$$wX^\lambda = X^{w\lambda}, \qquad \text{for } w \in W,\ \lambda \in P. \tag{3.4}$$

The ring of *symmetric functions* is

$$\mathbb{Z}[P]^W = \{f \in \mathbb{Z}[P] \mid wf = f \text{ for all } w \in W\}, \tag{3.5}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in W\lambda} X^\gamma, \qquad \text{for } \lambda \in P^+,$$

where $W\lambda$ is the orbit of $\lambda$ under the action of $W$. Then

$$\{m_\lambda \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[P]^W. \tag{3.6}$$

## 4. Type $SL_n(\mathbb{C})$

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the $\mathbb{R}$-basis of $\mathbb{R}^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{R}\}$ given by $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the $i$th entry. The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permuting the coordinates and, by restriction, $S_n$ acts on

$$\mathfrak{h}^*_\mathbb{R} = \{\gamma = \gamma_1\varepsilon_1 + \cdots + \gamma_n\varepsilon_n \mid \gamma_i \in \mathbb{R}, \gamma_1 + \cdots + \gamma_n = 0\}.$$

Let

$$\omega_n = \varepsilon_1 + \cdots + \varepsilon_n.$$

Then $S_n$ acts also on the $\mathbb{Z}$-submodule of $\mathfrak{h}^*_\mathbb{R}$ given by

$$\begin{aligned} P &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n - \tfrac{|\lambda|}{n}\omega_n \mid \lambda_i \in \mathbb{Z}_{\geq 0}\}. \\ \text{Let} \quad P^+ &= \{\lambda \in P \mid \lambda_1 \geq \cdots \geq \lambda_n\}, \\ \text{and} \quad P^{++} &= \{\lambda \in P \mid \lambda_1 > \cdots > \lambda_n\}. \end{aligned} \tag{4.1}$$

Then $P^+$ is a set of representatives of the orbits of the action of the natural action of $S_n$ on $P$. There is a bijection

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{where} \quad \rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} - \left(\tfrac{n-1}{2}\right)\omega_n. \tag{4.2}$$

Let

$$\mathbb{Z}[P] = \mathbb{Z}\text{-span}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in P. \tag{4.3}$$

For $1 \leq i \leq n$ write

$$x_i = X^{\varepsilon_i - \frac{1}{n}\omega_n} \qquad \text{so that} \qquad X^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \quad \text{for } \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n - \tfrac{|\lambda|}{n}\omega_n \in P.$$

Then $\mathbb{Z}[P]$ is the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by the ideal generated by the element $x_1 \cdots x_n - 1$,

$$\mathbb{Z}[P] = \frac{\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]}{\langle x_1 \cdots x_n - 1\rangle}.$$

The action of $S_n$ on $P$ induces an action of $S_n$ on $\mathbb{Z}[X]$ given by

$$wx_i = x_{w(i)}, \qquad \text{for } w \in S_n \text{ and } 1 \le i \le n, \tag{4.4}$$

and the ring of *symmetric functions* is

$$\mathbb{Z}[P]^{S_n} = \{f \in \mathbb{Z}[P] \mid wf = f \text{ for all } w \in S_n\}, \tag{4.5}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in S_n\lambda} X^\gamma, \qquad \text{for } \lambda \in P^+,$$

where $S_n\lambda$ is the orbit of $\lambda$ under the action of $S_n$. Then

$$\{m_\lambda \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[P]^{S_n}. \tag{4.6}$$

## 3. The path model

The path model of highest weight $\lambda$ is

$$B(\lambda) = \{f_{i_1} \dots f_{i_k} b_\lambda^+\},$$

the set of paths obtained by applying root operators to the highest weight path $b_\lambda^+$ in all possible ways.

There is an action of $W$ on $B(\lambda)$ given by flipping the $i$-strings in $B(\lambda)$.

Define

$$s_\lambda = \sum_{b \in B(\lambda)} X^{\mathrm{wt}(p)}.$$

**Proposition 3.1.** *Let $\lambda \in P^+$. Then*

$$\sum_{b \in B(\lambda)} X^{\mathrm{wt}(p)} \in \mathbb{Z}[P]^W.$$

*Proof.* If $p \in B(\lambda)$ and $w \in W$ then $wp \in B(\lambda)$ and $\mathrm{wt}(wp) = w\mathrm{wt}(p)$. ∎

**Theorem 3.2.** *Let $\lambda \in P^+$. Then*

$$s_\lambda = \sum_{b \in B(\lambda)} X^{\mathrm{wt}(p)}.$$

*Proof.*

∎

Another proof: The Demazure operator is

$$\tilde{T}_i f = \frac{f - s_i f}{1 - X^{-\alpha_i}}, \qquad \text{for } f \in \mathbb{Z}[X].$$

Then

$$\tilde{T}_i^2 = \tilde{T}_i, \qquad \text{and} \qquad T_i T_j T_i \cdots = T_j T_i T_j \cdots.$$

Furthermore

## Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.