Symmetric functions
Lecture Notes: Schur functions

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## 1. Skew polynomials

The polynomial ring

$$\mathbb{Z}[X_n] = \mathbb{Z}[x_1, \ldots, x_n] \qquad \text{has basis} \qquad \{x^\mu \mid \mu \in \mathbb{Z}_{\geq 0}^n\}, \quad \text{where} \quad x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n},$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$. The vector space of *skew polynomials* is

$$A_n = \{g \in \mathbb{Z}[x_1, \ldots, x_n] \mid wg = \det(w)g \text{ for all } w \in S_n\}.$$

If $f \in \mathbb{Z}[X_n]_n^S$ and $g \in A_n$ then $fg \in A_n$ and so $A_n$ is a $\mathbb{Z}[X_n]^{S_n}$-module.

The symmetric group $S_n$ acts on $\mathbb{Z}_{\geq 0}^n$ by permuting the coordinates. If

$$\begin{aligned} \mathbb{Z}^n &= \{(\gamma_1, \ldots, \gamma) \mid \gamma_i \in \mathbb{Z}\}, \\ P^+ &= \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}, \qquad \text{and} \\ P^{++} &= \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 > \gamma_2 > \cdots > \gamma_n\}, \end{aligned}$$

then $P^+$ is a set of representatives of the orbits of the $S_n$ action on $\mathbb{Z}^n$ and the map defined by

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array} \qquad \text{where} \qquad \rho = (n-1, n-2, \cdots, 2, 1, 0),$$

is a bijection.

Let

$$P_n^+ = \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0\}.$$

For each $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ such that $\mu_n \geq 0$,

$$a_\mu = \sum_{w \in S_n} \det(w) w x^\mu \tag{1.1}$$

is a skew polynomial. Since $a_\mu = \det(w) a_{w\mu}$ and $a_\mu = 0$ unless $\mu_1 > \mu_2 > \cdots > \mu_n$,

$$\{a_{\lambda+\rho} \mid \lambda \in P_n^+\} \qquad \text{is a basis of } A_n,$$

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and thus

$$A_n = \varepsilon \cdot \mathbb{Z}[X_n], \quad \text{where} \quad \varepsilon = \sum_{w \in S_n} \det(w) w.$$

The skew element

$$a_{\lambda+\rho} = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ & \vdots & \cdots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n} \end{pmatrix} \quad \text{is divisible by} \quad \prod_{n \ge j > i \ge 1} (x_j - x_i), \tag{1.2}$$

since the factors $(x_j - x_i)$ in the product on the right hand side are coprime in $\mathbb{Z}[x_1, \ldots, x_n]$ and setting $x_i = x_j$ makes the determinant vanish so that $a_{\lambda+\rho}$ must be divisible by $x_j - x_i$. When $\lambda = 0$, comparing coefficients of the maximal terms on each side shows that the *Vandermonde determinant*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^0 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^0 \\ \vdots & & \cdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^0 \end{pmatrix} = \prod_{n \ge j > i \ge 1} (x_j - x_i). \tag{1.3}$$

Since $\{a_{\lambda+\rho} \mid \lambda \in P_n^+\}$ is a basis of $A_n$, (???) shows that the inverse of the map

$$\begin{array}{ccc} \mathbb{Z}[x_1, \ldots, x_n]^{S_n} & \longrightarrow & A_n \\ f & \longmapsto & a_\rho f \end{array} \tag{1.4}$$

is well defined, and thus the map in (???) is an isomorphism of $\mathbb{Z}[X_n]^{S_n}$-modules.

The *Schur polynomials* are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P_n^+,$$

and since $\{a_{\lambda+\rho} \mid \lambda \in P_n^+\}$ is a basis of $A_n$ and the map in (???) is an isomorphism,

$$\{s_\lambda \mid \lambda \in P_n^+\} \qquad \text{is a basis of } \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

## 2. Schur functions

*Tableaux*

Let $\lambda$ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\ge 0}^n$ be a sequence of nonnegative integers. A *column strict tableau of shape* $\lambda$ *and weight* $\mu$ is a filling of the boxes of $\lambda$ with $\mu_1$ 1s, $\mu_2$ 2s, …, $\mu_n$ $n$s, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If $p$ is a column strict tableau write $\mathrm{shp}(p)$ and $\mathrm{wt}(p)$ for the shape and the weight of $p$ so that

$$\begin{aligned} \mathrm{shp}(p) &= (\lambda_1, \ldots, \lambda_n), \qquad \text{where} \quad \lambda_i = \text{number of boxes in row } i \text{ of } p, \quad \text{and} \\ \mathrm{wt}(p) &= (\mu_1, \ldots, \mu_n), \qquad \text{where} \quad \mu_i = \text{number of } i\text{s in } p. \end{aligned}$$

For example,

$$p = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
\hline
2 & 2 & 2 & 2 & 3 & 3 & 4 \\
\cline{1-7}
3 & 3 & 3 & 4 & 4 & 4 & 5 \\
\cline{1-7}
4 & 5 & 5 & 6 \\
\cline{1-4}
6 & 7 \\
\cline{1-2}
7 \\
\cline{1-1}
\end{array}
\qquad \text{has} \quad \begin{array}{l} \mathrm{shp}(p) = (9,7,7,4,2,1,0) \quad \text{and} \\ \mathrm{wt}(p) = (7,6,5,5,3,2,2). \end{array}$$

For a partition $\lambda$ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda \text{ and } \mathrm{wt}(p) = \mu\}, \end{aligned} \tag{2.1}$$

*Words*

Define $B_n = \{b_1, \ldots, b_n\}$ and let

$$B_n^{\otimes k} = \{b_{i_1} \cdots b_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$$

be the set of words of length $k$ in the alphabet $B$. For each $1 \leq i \leq n$ define *root operators*

$$\tilde{e}_i \colon B_n^{\otimes k} \longrightarrow B_n^{\otimes k} = \{0\} \qquad \text{and} \qquad \tilde{f}_i \colon B_n^{\otimes k} \longrightarrow B_n^{\otimes k} = \{0\}$$

by the following process. If $b = b_{i_1} \cdots b_{i_k}$ in $B_n^{\otimes k}$ place the value

$$\begin{aligned} +1 &\quad \text{over each } b_i, \\ -1 &\quad \text{over each } b_{i+1}, \\ 0 &\quad \text{over each } b_j,\ j \neq i, i+1. \end{aligned}$$

Ignoring 0s read this sequence of $\pm 1$ from left to right and successively remove adjacent $(+1, -1)$ pairs until the sequence is of the form

$$\underbrace{+1 \ \ +1 \ \ \ldots + \overset{\substack{\text{cogood} \\ \downarrow}}{1}}_{\text{conormal nodes}} \quad \underbrace{\overset{\substack{\text{good} \\ \downarrow}}{-1} \ \ -1 \ \ \ldots - 1}_{\text{normal nodes}}$$

The $-1$s in this sequence are the *normal nodes* and the $+1$s are the *conormal nodes.* The *good node* is the leftmost normal node and the *cogood node* is the right most conormal node. Then

$$\begin{aligned} \tilde{e}_i(b) &= \text{same as } b \text{ except with the cogood node path step changed to } b_i, \\ \tilde{f}_i(b) &= \text{same as } b \text{ except with the good node path step changed to } b_i, \end{aligned}$$

For example, if $n = 5$, $k = 30$,

$$b = b_4 b_3 b_3 b_1 b_2 b_2 b_4 b_4 b_1 b_2 b_3 b_3 b_2 b_1 b_1 b_2 b_3 b_3 b_2 b_1 b_4 b_5 b_5 b_1 b_1 b_1 b_1 b_2 b_2 b_4,$$

and $i=1$, then the parentheses in the table

$$\begin{array}{cccccccccccccccccccc}
 & & & ( & ) & -1 & & & ( & ) & & & -1 & ( & ( & ) & & & ) & +1 \\
0 & 0 & 0 & +1 & -1 & -1 & 0 & 0 & +1 & -1 & 0 & 0 & -1 & +1 & +1 & -1 & 0 & 0 & -1 & +1 \\
b_4 & b_3 & b_3 & b_1 & b_2 & b_2 & b_4 & b_4 & b_1 & b_2 & b_3 & b_3 & b_2 & b_1 & b_1 & b_2 & b_3 & b_3 & b_2 & b_1
\end{array}$$

$$\begin{array}{cccccccccc}
 & & & +1 & +1 & ( & ( & ) & ) & 0 \\
0 & 0 & 0 & +1 & +1 & +1 & +1 & -1 & -1 & 0 \\
b_4 & b_5 & b_5 & b_1 & b_1 & b_1 & b_1 & b_2 & b_2 & b_4
\end{array}$$

indicate the $(+1,-1)$ pairings and the numbers in the top row indicate the resulting sequence of $-1$s and $+1$s. Then

$$\begin{aligned}
\tilde{e}_1(b) &= b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3b_2b_1b_1b_2b_3b_3b_2\underline{b_2}b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4, \qquad \text{and} \\
\tilde{f}_1(b) &= b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3\underline{b_1}b_1b_1b_2b_3b_3b_2b_1b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4.
\end{aligned}$$

If $\lambda$ is a partition of $k$ define an imbedding

$$\begin{array}{ccc}
B(\lambda) & \longrightarrow & B_n^{\otimes k} \\
p & \longmapsto & b_{i_1}b_{i_2}\cdots b_{i_k}
\end{array}$$

where the entries $i_1i_2\cdots i_k$ are the entries of $p$ read in Arabic reading order. The action of $\tilde{e}_i$ and $\tilde{f}_i$ preserves the image of $B(\lambda)$ in $B_n^{\otimes k}$ and so the set $B(\lambda)$ can be viewed as a subcrystal.

If $B$ is a normal crystal and $b\in B$ the *$i$-string* of $b$ is the set

$$\tilde{f}_i^{\varphi_i(b)}b \overset{i}{\longleftrightarrow} \cdots \overset{i}{\longleftrightarrow} \tilde{f}_i^2 b \overset{i}{\longleftrightarrow} \tilde{f}_i b \overset{i}{\longleftrightarrow} b \overset{i}{\longleftrightarrow} \tilde{e}_i b \overset{i}{\longleftrightarrow} \tilde{e}_i^2 b \overset{i}{\longleftrightarrow} \cdots \overset{i}{\longleftrightarrow} \tilde{e}_i^{\varepsilon_i(b)}b,$$

and the extra condition for $B$ to be a normal crystal is equivalent to $\langle \mathrm{wt}(\tilde{e}_i^{\varepsilon_i(b)}b),\alpha_i^\vee\rangle = -\langle \mathrm{wt}(\tilde{f}_i^{\varphi(b)}b),\alpha_i^\vee\rangle$ so that *every $i$ string in a normal crystal $B$ is a model for a finite dimensional $\mathfrak{sl}_2$-module.*

If $B$ is a normal crystal define a bijection $s_i\colon B\to B$ by

$$s_i b = \begin{cases} \tilde{f}_i^{\langle \mathrm{wt}(b),\alpha_i^\vee\rangle}b, & \text{if } \langle \mathrm{wt}(b),\alpha_i^\vee\rangle \ge 0, \\ \tilde{e}_i^{-\langle \mathrm{wt}(b),\alpha_i^\vee\rangle}b, & \text{if } \langle \mathrm{wt}(b),\alpha_i^\vee\rangle \le 0, \end{cases} \qquad \text{so that} \qquad \mathrm{wt}(s_ib) = s_i\mathrm{wt}(b), \qquad \text{for all } b\in B.$$

The map $s_i$ flips each $i$-string in $B$.

**Proposition 2.2.** *[Kashiwara, Duke* **73** *(1994), 383-413] Let $B$ be a normal crystal. The maps $s_i\colon B\to B$ $i\in I$, define an action of $W$ on $B$.*

*Proof.* ∎

**Corollary 2.3.** *Let $B$ be a crystal. Then, for all $\mu\in P$ and $w\in W$,* $\mathrm{Card}(B_\mu)=\mathrm{Card}(B_{w\mu})$.

**Theorem 2.4.** *Let $B$ be a subcrystal of $\vec{B}$ such that $B_\mu$ is finite for all $\mu\in P$. Then*

$$\sum_{p\in B} e^{\mathrm{wt}(p)} = \sum_{\substack{b\in B \\ b\subseteq C-\rho}} s_{\mathrm{wt}(b)},$$

*where $s_\lambda$ denotes the Weyl character corresponding to $\lambda \in P^+$.*

*Proof.* Let $\chi^B$ be the sum on the left hand side. Then $\chi^B \in \mathbb{Z}[P]^W$ since the action of $W$ on $B$ defined in (???) satisfies $\mathrm{wt}(wp) = w\mathrm{wt}(p)$ for $w \in W$, $p \in B$. Thus

$$\sum_{p\in B} e^{\mathrm{wt}(p)} = \frac{1}{a_\rho}\chi^B a_\rho = \frac{1}{a_\rho}\chi^B \epsilon(e^\rho) = \frac{1}{a_\rho}\epsilon(\chi^B e^\rho)$$
$$= \frac{1}{a_\rho}\epsilon\left(\sum_{p\in B} e^p e^\rho\right) = \frac{1}{a_\rho}\sum_{p\in B} a_{\mathrm{wt}(p)+\rho} = \sum_{p\in B} s_{\mathrm{wt}(p)+\rho}.$$

Define a bijection

$$\tilde{:} B \to B \qquad \text{such that} \qquad s_{\mathrm{wt}(\tilde{p})} = -s_{\mathrm{wt}(p)},$$

for all $p \in B$ that are not highest weight.

Let $p \in B$. If $p$ is not highest weight then there is an $i$, $1 \le i \le n$, such that the last time $p$ leaves the region $C - \rho$ it leaves it by crossing the wall $H_{\alpha_i}$. We want

$$s_i \circ \mathrm{wt}(\tilde{p}) = \mathrm{wt}(p).$$

Since

$$s_i \circ \mathrm{wt}(p) = s_i(\mathrm{wt}(p) + \rho) - \rho = \mathrm{wt}(p) - \langle \mathrm{wt}(p), \alpha_i^\vee\rangle\alpha_i + \rho - \alpha_i - \rho = \mathrm{wt}(p) - \langle\langle \mathrm{wt}(p) + \rho, \alpha_i^\vee\rangle\rangle\alpha_i,$$

defining

$$\tilde{p} = \tilde{f}_i^{\langle \mathrm{wt}(p)+\rho, \alpha_i^\vee\rangle} p,$$

should do what we want. ?????? ∎

**Corollary 2.5.** *Let $\lambda \in P^+$. Then*

$$s_\lambda = \sum_{p\in B(\lambda)} e^{\mathrm{wt}(p)},$$

*so that $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$, where $K_{\lambda\mu} = \mathrm{Card}(B(\lambda)_\mu)$.*

*Cauchy kernels*

For a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$ of $k$ define

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots \qquad \text{so that} \qquad \frac{n!}{z_\lambda} = \mathrm{Card}(\{w \in S_k \mid w \text{ has cycle type } \lambda\}) \tag{2.6}$$

is the size of the conjugacy class indexed by $\lambda$ in the symmetric group $S_k$.

**Proposition 2.7.** *Let $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for a sequence for a sequence $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_{\ge 0}^n$.*

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda} = \sum_{\nu,\gamma} a_{\nu\gamma} x^\nu y^\gamma = \sum_\lambda s_\lambda(x) s_\lambda(y),$$

*where $a_{\nu\gamma}$ is the set of matrices with entries in $\mathbb{Z}_{\geq 0}$ with row sums $\nu$ and column sums $\gamma$.*

*Proof.* (a)

$$\prod_j \prod_i \frac{1}{1-x_iy_j} = \prod_j \left( \sum_{k_j\in\mathbb{Z}_{\geq 0}} h_{j_k}(x) y_j^{k_j} \right)$$
$$= \sum_{k_1,k_2,\ldots} (h_{k_1}(x)h_{k_2}(x)\cdots)(y_1^{k_1}y_2^{k_2}\cdots) = \sum_\lambda h_\lambda(x)m_\lambda(y).$$

(b) Recalling that

$$\ln(1-x_iy_j) = \sum_{k\geq 1} \frac{x_i^k y_j^k}{k} \quad \text{since} \quad \ln(1-t) = \int \frac{1}{1-t}dt = \int (1+t+t^2+\cdots)dt,$$

we have

$$\prod_{i,j}\frac{1}{1-x_iy_j} = \exp\ln\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right) = \exp\left(\sum_{i,j}\ln(1-x_iy_j)\right) = \exp\left(\sum_k\sum_{i,j}\frac{x_i^ky_j^k}{k}\right)$$
$$= \exp\left(\sum_k \frac{p_k(x)p_k(y)}{k}\right) = \prod_k \exp\left(\frac{p_k(x)p_k(y)}{k}\right) = \prod_k \sum_{m_k\geq 0}\left(\frac{p_k^{m_k}(x)p_k^{m_k}(y)}{k^{m_k}m_k!}\right)$$
$$= \sum_{m_1,m_2,\ldots}\left(\frac{p_1^{m_1}(x)p_2^{m_2}(x)\cdots p_1^{m_1}(y)p_2^{m_2}(y)\cdots}{1^{m_1}m_1!2^{m_2}m_2!\cdots}\right) = \sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}$$

(c) Let $A$ be the set of matrices with rows and columns indexed by $\mathbb{Z}_{>0}$ and with entries from $\mathbb{Z}_{\geq 0}$. Then

$$\prod_{i,j}\frac{1}{1-x_iy_j} = \left(\sum_{a_{11}\in\mathbb{Z}_{\geq 0}}(x_1y_1)^{a_{11}}\right)\left(\sum_{a_{12}\in\mathbb{Z}_{\geq 0}}(x_1y_2)^{a_{12}}\right)\cdots = \sum_{a\in A}\prod_{i,j}(x_iy_j)^{a_{ij}} = \sum_{\mu,\gamma} a_{\nu\gamma}x^\nu y^\gamma.$$

(d) Let $B = \bigsqcup_\lambda B(\lambda)$ be the set of column strict tableaux.

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda \sum_{P\in B(\lambda)}\sum_{Q\in B(\lambda)} x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)} = \sum_{\substack{P,Q\in B\\ \mathrm{sh}(P)=\mathrm{sh}(Q)}} x^{\mathrm{wt}(P)}y^{wt(Q)}.$$

∎

## Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.