Symmetric functions
Lecture Notes: Schur functions

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1. Skew polynomials

The polynomial ring

$$\mathbb{Z}[X_n] = \mathbb{Z}[x_1, \ldots, x_n] \qquad \text{has basis} \qquad \{x^\mu \mid \mu \in \mathbb{Z}^n_{\geq 0}\}, \quad \text{where} \quad x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n},$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$. The vector space of *skew polynomials* is

$$A_n = \{g \in \mathbb{Z}[x_1, \ldots, x_n] \mid wg = \det(w)g \text{ for all } w \in S_n\}.$$

If $f \in \mathbb{Z}[X_n]^S_n$ and $g \in A_n$ then $fg \in A_n$ and so A_n is a $\mathbb{Z}[X_n]^{S_n}$-module.

The symmetric group S_n acts on $\mathbb{Z}^n_{\geq 0}$ by permuting the coordinates. If

$$\begin{aligned} \mathbb{Z}^n &= \{(\gamma_1, \ldots, \gamma) \mid \gamma_i \in \mathbb{Z}\}, \\ P^+ &= \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}, \qquad \text{and} \\ P^{++} &= \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 > \gamma_2 > \cdots > \gamma_n\}, \end{aligned}$$

then P^+ is a set of representatives of the orbits of the S_n action on $\mathbb{Z}^n$ and the map defined by

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array} \qquad \text{where} \qquad \rho = (n-1, n-2, \cdots, 2, 1, 0),$$

is a bijection.

Let

$$P^+_n = \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0\}.$$

For each $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ such that $\mu_n \geq 0$,

$$a_\mu = \sum_{w \in S_n} \det(w) w x^\mu \tag{1.1}$$

is a skew polynomial. Since $a_\mu = \det(w) a_{w\mu}$ and $a_\mu = 0$ unless $\mu_1 > \mu_2 > \cdots > \mu_n$,

$$\{a_{\lambda+\rho} \mid \lambda \in P^+_n\} \qquad \text{is a basis of } A_n,$$

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and thus

$$A_n = \varepsilon \cdot \mathbb{Z}[X_n], \quad \text{where} \quad \varepsilon = \sum_{w \in S_n} \det(w) w.$$

The skew element

$$a_{\lambda+\rho} = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ & \vdots & \cdots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n} \end{pmatrix} \quad \text{is divisible by} \quad \prod_{n \geq j > i \geq 1} (x_j - x_i), \tag{1.2}$$

since the factors $(x_j - x_i)$ in the product on the right hand side are coprime in $\mathbb{Z}[x_1, \ldots, x_n]$ and setting $x_i = x_j$ makes the determinant vanish so that $a_{\lambda+\rho}$ must be divisible by $x_j - x_i$. When $\lambda = 0$, comparing coefficients of the maximal terms on each side shows that the *Vandermonde determinant*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^0 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^0 \\ \vdots & & \cdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^0 \end{pmatrix} = \prod_{n \geq j > i \geq 1} (x_j - x_i). \tag{1.3}$$

Since $\{a_{\lambda+\rho} \mid \lambda \in P_n^+\}$ is a basis of A_n, (???) shows that the inverse of the map

$$\begin{array}{ccc} \mathbb{Z}[x_1, \ldots, x_n]^{S_n} & \longrightarrow & A_n \\ f & \longmapsto & a_\rho f \end{array} \tag{1.4}$$

is well defined, and thus the map in (???) is an isomorphism of $\mathbb{Z}[X_n]^{S_n}$-modules.

The *Schur polynomials* are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P_n^+,$$

and since $\{a_{\lambda+\rho} \mid \lambda \in P_n^+\}$ is a basis of A_n and the map in (???) is an isomorphism,

$$\{s_\lambda \mid \lambda \in P_n^+\} \qquad \text{is a basis of } \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

2. Schur functions

Tableaux

Let λ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ be a sequence of nonnegative integers. A *column strict tableau of shape* λ *and weight* μ is a filling of the boxes of λ with μ_1 1s, μ_2 2s, ..., μ_n ns, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If p is a column strict tableau write $\mathrm{shp}(p)$ and $\mathrm{wt}(p)$ for the shape and the weight of p so that

$$\begin{aligned} \mathrm{shp}(p) &= (\lambda_1, \ldots, \lambda_n), && \text{where} \quad \lambda_i = \text{number of boxes in row } i \text{ of } p, \quad \text{and} \\ \mathrm{wt}(p) &= (\mu_1, \ldots, \mu_n), && \text{where} \quad \mu_i = \text{number of } i\text{s in } p. \end{aligned}$$

For example,

$$p = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2\\ \hline 2&2&2&2&3&3&4 \\ \cline{1-7} 3&3&3&4&4&4&5 \\ \cline{1-7} 4&5&5&6 \\ \cline{1-4} 6&7 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \qquad \text{has} \quad \begin{array}{l}\mathrm{shp}(p) = (9,7,7,4,2,1,0) \quad \text{and} \\ \mathrm{wt}(p) = (7,6,5,5,3,2,2).\end{array}$$

For a partition λ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda \text{ and } \mathrm{wt}(p) = \mu\}, \end{aligned} \tag{2.1}$$

Words

Define $B_n = \{b_1, \ldots, b_n\}$ and let

$$B_n^{\otimes k} = \{b_{i_1} \cdots b_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$$

be the set of words of length k in the alphabet B. For each $1 \leq i \leq n$ define *root operators*

$$\tilde{e}_i \colon B_n^{\otimes k} \longrightarrow B_n^{\otimes k} = \{0\} \qquad \text{and} \qquad \tilde{f}_i \colon B_n^{\otimes k} \longrightarrow B_n^{\otimes k} = \{0\}$$

by the following process. If $b = b_{i_1} \cdots b_{i_k}$ in $B_n^{\otimes k}$ place the value

$$\begin{array}{rl} +1 & \text{over each } b_i, \\ -1 & \text{over each } b_{i+1}, \\ 0 & \text{over each } b_j,\ j \neq i, i+1. \end{array}$$

Ignoring 0s read this sequence of ± 1 from left to right and successively remove adjacent $(+1, -1)$ pairs until the sequence is of the form

$$\underbrace{+1\ \ +1\ \ \ldots + \overset{\substack{\text{cogood}\\ \downarrow}}{1}}_{\text{conormal nodes}} \quad \underbrace{\overset{\substack{\text{good}\\ \downarrow}}{-1}\ \ -1\ \ \ldots - 1}_{\text{normal nodes}}$$

The -1s in this sequence are the *normal nodes* and the $+1$s are the *conormal nodes*. The *good node* is the leftmost normal node and the *cogood node* is the right most conormal node. Then

$$\begin{aligned} \tilde{e}_i(b) &= \text{same as } b \text{ except with the cogood node path step changed to } b_i, \\ \tilde{f}_i(b) &= \text{same as } b \text{ except with the good node path step changed to } b_i, \end{aligned}$$

For example, if $n = 5$, $k = 30$,

$$b = b_4 b_3 b_3 b_1 b_2 b_2 b_4 b_4 b_1 b_2 b_3 b_3 b_2 b_1 b_1 b_2 b_3 b_3 b_2 b_1 b_4 b_5 b_5 b_1 b_1 b_1 b_1 b_2 b_2 b_4,$$

and $i=1$, then the parentheses in the table

			(	)	-1			(	)			-1	(	(	)			)	$+1$
0	0	0	$+1$	-1	-1	0	0	$+1$	-1	0	0	-1	$+1$	$+1$	-1	0	0	-1	$+1$
b_4	b_3	b_3	b_1	b_2	b_2	b_4	b_4	b_1	b_2	b_3	b_3	b_2	b_1	b_1	b_2	b_3	b_3	b_2	b_1

			$+1$	$+1$	(	(	)	)	0
0	0	0	$+1$	$+1$	$+1$	$+1$	-1	-1	0
b_4	b_5	b_5	b_1	b_1	b_1	b_1	b_2	b_2	b_4

indicate the $(+1,-1)$ pairings and the numbers in the top row indicate the resulting sequence of -1s and $+1$s. Then

$$\tilde e_1(b) = b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3b_2b_1b_1b_2b_3b_3b_2\underline{b_2}b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4, \qquad \text{and}$$
$$\tilde f_1(b) = b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3\underline{b_1}b_1b_1b_2b_3b_3b_2b_1b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4.$$

If λ is a partition of k define an imbedding

$$\begin{matrix} B(\lambda) & \longrightarrow & B_n^{\otimes k} \\ p & \longmapsto & b_{i_1}b_{i_2}\cdots b_{i_k} \end{matrix}$$

where the entries $i_1i_2\cdots i_k$ are the entries of p read in Arabic reading order. The action of $\tilde e_i$ and $\tilde f_i$ preserves the image of $B(\lambda)$ in $B_n^{\otimes k}$ and so the set $B(\lambda)$ can be viewed as a subcrystal.

If B is a normal crystal and $b\in B$ the *i-string* of b is the set

$$\tilde f_i^{\varphi_i(b)}b \overset{i}{\longleftrightarrow} \cdots \overset{i}{\longleftrightarrow} \tilde f_i^2 b \overset{i}{\longleftrightarrow} \tilde f_i b \overset{i}{\longleftrightarrow} b \overset{i}{\longleftrightarrow} \tilde e_i b \overset{i}{\longleftrightarrow} \tilde e_i^2 b \overset{i}{\longleftrightarrow} \cdots \overset{i}{\longleftrightarrow} \tilde e_i^{\varepsilon_i(b)} b,$$

and the extra condition for B to be a normal crystal is equivalent to $\langle \mathrm{wt}(\tilde e_i^{\varepsilon_i(b)}b),\alpha_i^\vee\rangle = -\langle \mathrm{wt}(\tilde f_i^{\varphi(b)}b),\alpha_i^\vee\rangle$ so that *every i string in a normal crystal B is a model for a finite dimensional $\mathfrak{sl}_2$-module.*

If B is a normal crystal define a bijection $s_i\colon B\to B$ by

$$s_i b = \begin{cases} \tilde f_i^{\langle \mathrm{wt}(b),\alpha_i^\vee\rangle} b, & \text{if } \langle \mathrm{wt}(b),\alpha_i^\vee\rangle \ge 0, \\ \tilde e_i^{-\langle \mathrm{wt}(b),\alpha_i^\vee\rangle} b, & \text{if } \langle \mathrm{wt}(b),\alpha_i^\vee\rangle \le 0, \end{cases} \qquad \text{so that} \qquad \mathrm{wt}(s_ib) = s_i\mathrm{wt}(b), \qquad \text{for all } b\in B.$$

The map s_i flips each i-string in B.

Proposition 2.2. *[Kashiwara, Duke* **73** *(1994), 383-413] Let B be a normal crystal. The maps $s_i\colon B\to B$ $i\in I$, define an action of W on B.*

Proof. ∎

Corollary 2.3. *Let B be a crystal. Then, for all $\mu\in P$ and $w\in W$,* $\mathrm{Card}(B_\mu) = \mathrm{Card}(B_{w\mu})$.

Theorem 2.4. *Let B be a subcrystal of $\vec B$ such that B_μ is finite for all $\mu\in P$. Then*

$$\sum_{p\in B} e^{\mathrm{wt}(p)} = \sum_{\substack{b\in B\\ b\subseteq C-\rho}} s_{\mathrm{wt}(b)},$$

where s_λ denotes the Weyl character corresponding to $\lambda \in P^+$.

Proof. Let χ^B be the sum on the left hand side. Then $\chi^B \in \mathbb{Z}[P]^W$ since the action of W on B defined in (???) satisfies $\mathrm{wt}(wp) = w\mathrm{wt}(p)$ for $w \in W$, $p \in B$. Thus

$$\sum_{p\in B} e^{\mathrm{wt}(p)} = \frac{1}{a_\rho}\chi^B a_\rho = \frac{1}{a_\rho}\chi^B \epsilon(e^\rho) = \frac{1}{a_\rho}\epsilon(\chi^B e^\rho)$$
$$= \frac{1}{a_\rho}\epsilon\left(\sum_{p\in B} e^p e^\rho\right) = \frac{1}{a_\rho}\sum_{p\in B} a_{\mathrm{wt}(p)+\rho} = \sum_{p\in B} s_{\mathrm{wt}(p)+\rho}.$$

Define a bijection

$$\tilde{:} B \to B \qquad \text{such that} \qquad s_{\mathrm{wt}(\tilde{p})} = -s_{\mathrm{wt}(p)},$$

for all $p \in B$ that are not highest weight.

Let $p \in B$. If p is not highest weight then there is an i, $1 \le i \le n$, such that the last time p leaves the region $C - \rho$ it leaves it by crossing the wall H_{α_i}. We want

$$s_i \circ \mathrm{wt}(\tilde{p}) = \mathrm{wt}(p).$$

Since

$$s_i \circ \mathrm{wt}(p) = s_i(\mathrm{wt}(p)+\rho) - \rho = \mathrm{wt}(p) - \langle \mathrm{wt}(p), \alpha_i^\vee\rangle \alpha_i + \rho - \alpha_i - \rho = \mathrm{wt}(p) - \langle\langle \mathrm{wt}(p)+\rho, \alpha_i^\vee\rangle\rangle \alpha_i,$$

defining

$$\tilde{p} = \tilde{f}_i^{\langle \mathrm{wt}(p)+\rho, \alpha_i^\vee\rangle} p,$$

should do what we want. ?????? ∎

Corollary 2.5. *Let $\lambda \in P^+$. Then*

$$s_\lambda = \sum_{p\in B(\lambda)} e^{\mathrm{wt}(p)},$$

so that $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$, where $K_{\lambda\mu} = \mathrm{Card}(B(\lambda)_\mu)$.

Cauchy kernels

For a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$ of k define

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots \qquad \text{so that} \qquad \frac{n!}{z_\lambda} = \mathrm{Card}(\{w \in S_k \mid w \text{ has cycle type } \lambda\}) \tag{2.6}$$

is the size of the conjugacy class indexed by λ in the symmetric group S_k.

Proposition 2.7. *Let $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for a sequence for a sequence $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n_{\ge 0}$.*

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda} = \sum_{\nu,\gamma} a_{\nu\gamma} x^\nu y^\gamma = \sum_\lambda s_\lambda(x) s_\lambda(y),$$

where $a_{\nu\gamma}$ is the set of matrices with entries in $\mathbb{Z}_{\geq 0}$ with row sums ν and column sums γ.

Proof. (a)

$$\prod_j \prod_i \frac{1}{1-x_iy_j} = \prod_j \left(\sum_{k_j \in \mathbb{Z}_{\geq 0}} h_{j_k}(x) y_j^{k_j} \right)$$
$$= \sum_{k_1,k_2,\ldots} (h_{k_1}(x)h_{k_2}(x)\cdots)(y_1^{k_1}y_2^{k_2}\cdots) = \sum_\lambda h_\lambda(x)m_\lambda(y).$$

(b) Recalling that

$$\ln(1-x_iy_j) = \sum_{k\geq 1} \frac{x_i^k y_j^k}{k} \quad \text{since} \quad \ln(1-t) = \int \frac{1}{1-t}dt = \int (1+t+t^2+\cdots)dt,$$

we have

$$\prod_{i,j} \frac{1}{1-x_iy_j} = \exp\ln\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right) = \exp\left(\sum_{i,j}\ln(1-x_iy_j)\right) = \exp\left(\sum_k\sum_{i,j}\frac{x_i^ky_j^k}{k}\right)$$
$$= \exp\left(\sum_k \frac{p_k(x)p_k(y)}{k}\right) = \prod_k \exp\left(\frac{p_k(x)p_k(y)}{k}\right) = \prod_k \sum_{m_k\geq 0}\left(\frac{p_k^{m_k}(x)p_k^{m_k}(y)}{k^{m_k}m_k!}\right)$$
$$= \sum_{m_1,m_2,\ldots}\left(\frac{p_1^{m_1}(x)p_2^{m_2}(x)\cdots p_1^{m_1}(y)p_2^{m_2}(y)\cdots}{1^{m_1}m_1!2^{m_2}m_2!\cdots}\right) = \sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}$$

(c) Let A be the set of matrices with rows and columns indexed by $\mathbb{Z}_{>0}$ and with entries from $\mathbb{Z}_{\geq 0}$. Then

$$\prod_{i,j}\frac{1}{1-x_iy_j} = \left(\sum_{a_{11}\in\mathbb{Z}_{\geq 0}} (x_1y_1)^{a_{11}}\right)\left(\sum_{a_{12}\in\mathbb{Z}_{\geq 0}} (x_1y_2)^{a_{12}}\right)\cdots = \sum_{a\in A}\prod_{i,j}(x_iy_j)^{a_{ij}} = \sum_{\mu,\gamma} a_{\nu\gamma}x^\nu y^\gamma.$$

(d) Let $B = \bigsqcup_\lambda B(\lambda)$ be the set of column strict tableaux.

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda \sum_{P\in B(\lambda)}\sum_{Q\in B(\lambda)} x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)} = \sum_{\substack{P,Q\in B \\ \mathrm{sh}(P)=\mathrm{sh}(Q)}} x^{\mathrm{wt}(P)}y^{wt(Q)}.$$

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Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.