Symmetric functions
Lecture Notes

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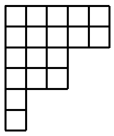
1. Symmetric functions

Partitions

A *partition* is a collection μ of boxes in a corner where the convention is that gravity goes up and to the left. As for matrices, the rows and columns of μ are indexed from top to bottom and left to right, respectively.

$$\begin{array}{ll} \text{The } \textit{parts} \text{ of } \mu \text{ are} & \mu_i = (\text{the number of boxes in row } i \text{ of } \mu), \\ \text{the } \textit{length} \text{ of } \mu \text{ is} & \ell(\mu) = (\text{the number of rows of } \mu), \\ \text{the } \textit{size} \text{ of } \mu \text{ is} & |\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)} = (\text{the number of boxes of } \mu). \end{array} \tag{1.1}$$

Then μ is determined by (and identified with) the sequence $\mu = (\mu_1, \ldots, \mu_\ell)$ of positive integers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0$, where $\ell = \ell(\mu)$. For example,

$(5,5,3,3,1,1) =$  .

A *partition of* k is a partition λ with k boxes. Write $\lambda \vdash k$ if λ is a partition of k. Make the convention that $\lambda_i = 0$ if $i > \ell(\lambda)$. The *dominance order* is the partial order on the set of partitions of k,

$$P^+(k) = \{\text{partitions of } k\} = \{\lambda = (\lambda_1, \ldots, \lambda_\ell) \mid \lambda_1 \geq \cdots \geq \lambda_\ell > 0,\ \lambda_1 + \ldots + \lambda_\ell = k\},$$

given by

$$\lambda \geq \mu \qquad \text{if} \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for all } 1 \leq i \leq \max\{\ell(\lambda), \ell(\mu)\}.$$

PUT THE PICTURE OF THE HASSE DIAGRAM FOR $k = 6$ HERE.

Tableaux

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Let λ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ be a sequence of nonnegative integers. A *column strict tableau of shape* λ *and weight* μ is a filling of the boxes of λ with μ_1 1s, μ_2 2s, …, μ_n ns, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If p is a column strict tableau write $\mathrm{shp}(p)$ and $\mathrm{wt}(p)$ for the shape and the weight of p so that

$$\begin{aligned} \mathrm{shp}(p) &= (\lambda_1, \ldots, \lambda_n), &\quad \text{where} \quad \lambda_i &= \text{number of boxes in row } i \text{ of } p, \quad \text{and} \\ \mathrm{wt}(p) &= (\mu_1, \ldots, \mu_n), &\quad \text{where} \quad \mu_i &= \text{number of } i\,\text{s in } p. \end{aligned}$$

For example,

$$p = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1&1&1&1&1&1&1&2&2 \\ \hline 2&2&2&2&3&3&4 \\ \cline{1-7} 3&3&3&4&4&4&5 \\ \cline{1-7} 4&5&5&6 \\ \cline{1-4} 6&7 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \text{has} \quad \begin{aligned} \mathrm{shp}(p) &= (9,7,7,4,2,1,0) \quad \text{and} \\ \mathrm{wt}(p) &= (7,6,5,5,3,2,2). \end{aligned}$$

For a partition λ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda \text{ and } \mathrm{wt}(p) = \mu\}, \end{aligned} \tag{1.2}$$

Symmetric functions

The symmetric group S_n acts on the vector space

$$\mathbb{Z}^n = \mathbb{Z}\text{-span}\{x_1, \ldots, x_n\} \qquad \text{by} \qquad w x_i = x_{w(i)},$$

for $w \in S_n$, $1 \leq i \leq n$. This action induces an action of S_n on the polynomial ring $\mathbb{Z}[X_n] = \mathbb{Z}[x_1, \ldots, x_n]$ by ring automorphisms. For a sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ of nonnegative integers let

$$x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}, \qquad \text{so that} \quad \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}\text{-span}\{x^\gamma \mid \gamma \in \mathbb{Z}_{\geq 0}^n\}.$$

The ring of *symmetric functions* is

$$\mathbb{Z}[X_n]^{S_n} = \{f \in \mathbb{Z}[X_n] \mid wf = f \text{ for all } w \in S_n\}, \tag{1.3}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma, \qquad \text{for } \lambda \in \mathbb{Z}_{\geq 0}^n,$$

where $S_n\lambda$ is the orbit of λ under the action of S_n. Let

$$P^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\} \tag{1.4}$$

so that

$$\{m_\lambda \mid \lambda \in P^+\} \quad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[X_n]^{S_n}. \tag{1.5}$$

Interpolating symmetric functions

Define $q_r(X_n; q, t) = q_r(x_1, \ldots, x_n; q, t)$ by the generating function

$$\prod_{i=1}^n \frac{1 - x_i tz}{1 - x_i qz} = (q-t) \sum_{r \geq 0} q_r(x_1, \ldots, x_n; q, t) z^r.$$

Note that with this definition $q_0 = 1/(q-t)$. Define the *elementary symmetric functions*, the *complete symmetric functions* and the *power symmetric functions* by the formulas

$$\begin{aligned} (-t)^{r-1} e_r(X_n) &= q_r(X_n; 0, t), \\ q^{r-1} h_r(X_n) &= q_r(X_n; q, 0), \quad \text{and} \\ q^{r-1} p_r(X_n) &= q_r(X_n; q, q), \quad \text{respectively.} \end{aligned} \tag{1.6}$$

The elementary symmetric functions have special importance because of the following ways in which they appear naturally.

(1) If $f(t)$ is a polynomial in t with roots $\gamma_1, \ldots, \gamma_n$ then

$$\text{the coefficient of } t^r \text{ in } f(t) \text{ is } (-1)^{n-r} e_r(\gamma_1, \ldots, \gamma_n). \tag{1.7}$$

(2) If A is an $n \times n$ matrix with entries in $\mathbb{F}$ with eigenvalues $\gamma_1, \ldots, \gamma_n$ then the trace of the action of A on the r^{th} exterior power of the vector space $\mathbb{F}^n$ is

$$\begin{aligned} &\mathrm{tr}(A, \textstyle\bigwedge^r \mathbb{F}^n) = e_r(\gamma_1, \ldots, \gamma_n), \quad \text{so that} \\ &\mathrm{Tr}(A) = e_1(\gamma_1, \ldots, \gamma_n), \qquad \text{and} \qquad \det(A) = e_n(\gamma_1, \ldots, \gamma_n), \end{aligned} \tag{1.8}$$

and the characteristic polynomial of A is

$$\mathrm{char}_t(A) = \sum_{r=0}^n (-1)^{n-r} e_{n-r}(\gamma_1, \ldots, \gamma_n) t^r. \tag{1.9}$$

Expanding $\displaystyle\frac{1 - x_i tz}{1 - x_i qz} = 1 + (q-t) \sum_{\ell > 0} q^{\ell-1} x_i^\ell z^\ell$ and multiplying out

$$\prod_{i=1}^n \frac{1 - x_i tz}{1 - x_i qz} = \frac{1 - x_1 tz}{1 - x_1 qz} \cdots \frac{1 - x_n tz}{1 - x_n qz}$$

gives

$$q_r = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} (q-t)^{\mathrm{Card}(\{j \mid i_j < i_{j+1}\})} q^{\mathrm{Card}(\{j \mid i_j = i_{j+1}\})} x_{i_1} x_{i_2} \cdots x_{i_r}. \tag{1.10}$$

from which it follows that

$$q_r = \sum_{\lambda \vdash r} (q-t)^{\ell(\lambda)-1} q^{r-\ell(\lambda)} m_\lambda(x_1, \ldots, x_n).$$

For an $n \times n$ matrix $a = (a_{ij})$ with entries from $\mathbb{Z}_{\geq 0}$ let

$$x^a = \prod_{i=1}^{n} (x_i)^{a_{ij}}, \qquad \text{and} \qquad \mathrm{wt}(a) = q^{|\lambda| - \ell(a)}(q-t)^{\ell(a)-\ell(\lambda)},$$

where $\ell(a)$ is the number of nonzero entries in a, $\ell(\lambda)$ is the number of nonzero entries in λ, and $|\lambda|$ is the sum of the entries of λ. Define

$$\begin{matrix} rs(a) = (\rho_1, \ldots, \rho_n), \\ cs(a) = (\gamma_1, \ldots, \gamma_n), \end{matrix} \qquad \text{where} \qquad \rho_i = \sum_{j=1}^{\ell} a_{ij} \quad \text{and} \quad \gamma_j = \sum_{i=1}^{n} a_{ij},$$

so that $rs(a)$ and $cs(a)$ are the sequences of row sums and column sums of a, respectively.

Proposition 1.11. *For a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ define*

$$q_\lambda(X_n; q, t) = q_{\lambda_1}(X_n; q, t) \cdot q_{\lambda_2}(X_n; q, t) \cdots q_{\lambda_\ell}(X_n; q, t).$$

Then

$$q_\lambda = \sum_\mu a_{\lambda\mu}(q,t) m_\mu, \qquad \text{where} \qquad a_{\lambda\mu}(q,t) = \sum_{a \in A_{\lambda\mu}} \mathrm{wt}(a),$$

and the sum is over partitions μ such that $|\mu = |\lambda|$.

Proof. If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ then

$$q_\lambda = \prod_{j=1}^{\ell} q_{\lambda_j} = \sum_{rs(a)=\lambda} \mathrm{wt}(a) x^a = \sum_{\gamma \in \mathbb{Z}_{\geq 0}^n} \sum_{\substack{rs(a)=\lambda \\ cs(a)=\gamma}} \mathrm{wt}(a) x^\gamma = \sum_\mu a_{\lambda\mu}(q,t) m_\mu.$$

∎

Multiplying out

$$\prod_{i=1}^{n} \frac{1 - x_i t z}{1 - x_i q z} = \frac{1}{1 - x_1 q z} \cdot \frac{1}{1 - x_2 q z} \cdots \frac{1}{1 - x_n q z}(1 - x_n t z)(1 - x_{n-1} t z) \cdots (1 - x_1 t z)$$

gives

$$q_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k > i_{k+1} > \cdots > i_r} q^{k-1}(-t)^{r-k} x_{i_1} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_r}. \tag{1.12}$$

The bijection ???? between sequences $i_1 \leq i_2 \leq \cdots \leq i_k > i_{k+1} > \cdots > i_r$ and column strict tableaux of shape $(k1^{r-k})$ yields

$$q_r = \sum_{k=1}^{r} (-t)^{r-k} q^{k-1} s_{(k1^{r-k})}(X_n). \tag{1.13}$$

For each positive integer k define

$$[k]_{q,t} = \frac{q^k - t^k}{q - t} = q^{k-1} + t q^{k-2} + \cdots + t^{k-2} q + t^{k-1}.$$

Comparing coefficients of z^r on each side of

$$\left(\prod_{i=1}^{n} \frac{1-x_i s z}{1-x_i t z}\right)\left(\prod_{i=1}^{n} \frac{1-x_i t z}{1-x_i q z}\right) = \prod_{i=1}^{n} \frac{1-x_i s z}{1-x_i q z}$$

gives

$$\begin{aligned}(t-s)q_r(X_n;t,s) + (q-t)(t-s)\left(\sum_{j=1}^{r-1} q_j(X_n;q,t)q_{r-j}(X_n;t,s)\right) \\ + (q-t)q_r(X_n;q,t) = (q-s)q_r(X_n;q,s).\end{aligned} \tag{1.14}$$

Example. Putting $s=0$, $q=0$ and $t=0$ in (???) yield, respectively,

$$\begin{aligned}
&q_r(X_n;q,t) + \left(\sum_{j=1}^{r-1} h_j(X_n)t^j q_{r-j}(X_n;q,t)\right) - h_r(X_n)[r]_{q,t} = 0 \\
&q_r(X_n;q,t) + \left(\sum_{j=1}^{r-1} e_j(X_n)(-q)^j q_{r-j}(X_n;q,t)\right) + (-1)^r e_r(X_n)[r]_{q,t} = 0 \\
&\sum_{j=0}^{r} (-t)^{r-j}[j]_{q,t} h_j(X_n)e_{r-j}(X_n) = (q-t)q_r(X_n;q,t).
\end{aligned}$$

Then putting????? (???)gives

$$rq_r(X_n;q,t) - \left(\sum_{j=1}^{r-1} p_j(X_n)(q^j - t^j)q_{r-j}(X_n;q,t)\right) - p_r(X_n)[r]_{q,t} = 0.$$

and the *Newton identities*

$$kh_k = \sum_{i=1} p_i h_{k-i} \qquad \text{and} \qquad ke_k = \sum_{i=1}^{k} (-1)^{i-1} p_i e_{k-i},$$

are obtained by putting??? in (???).

Proposition 1.15. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. Then*

(a) $e_{\lambda'} = \sum_{\mu \le \lambda} a_{\lambda'\mu} m_\mu$,
where $a_{\lambda\mu}$ is the number of matrices with entries from $\{0,1\}$ with row sums λ' and column sums μ.

(b) $a_{\lambda'\lambda} = 1$ *and* $a_{\lambda'\mu} = 0$ *unless* $\mu \le \lambda$.

(c) $\{e_\lambda \mid \ell(\lambda') \le n\}$ *is a $\mathbb{Z}$-basis of* $\mathbb{Z}[X_n]^{S_n}$.

(d) $\mathbb{Z}[x_1, \ldots, x_n]^{S_n} = \mathbb{Z}[e_1, \ldots, e_n] = \mathbb{Z}[h_1, \ldots, h_n]$ *and* $\mathbb{Q}[x_1, \ldots, x_n]^{S_n} = \mathbb{Q}[p_1, \ldots, p_n]$.

(e) *The set* $\{h_\lambda \mid \ell(\lambda') \le n\}$ *is a basis of* $\mathbb{Z}[X_n]^{S_n}$.

Proof. (a) follows by putting $q = 0$ in (???).

(b) Since there is a unique matrix A with $rs(A) = \lambda'$ and $cs(A) = \lambda$, $a_{\lambda'\lambda} = 1$. If A is a $0,1$ matrix with $rs(A) = \lambda'$ and $cs(A) = \mu$ then $\mu_1 + \cdots + \mu_i \le \lambda_1 + \cdots + \lambda_i$ since there are at most $\lambda_1 + \cdots + \lambda_i$ nonzero entries in the first i columns of A. Thus $a_{\lambda'\mu} = 0$ unless $\mu \le \lambda$.

(c) is a consequence of (b) and the fact that $\{m_\lambda \mid \ell(\lambda) \le n\}$ is a basis of $\mathbb{Z}[X_n]^{S_n}$.

(d) The first equality is an immediate consequence of (c). The second equality follows from the identity (???), which allows one to, inductively, expand h_r in terms of $e_r, e_{r-1}, \ldots, e_1$. Similarly, the third equality follows from the Newton identity (???) which allows one to, inductively, expand p_r in terms of $e_r, e_{r-1}, \ldots, e_1$ (with coefficients in $\mathbb{Q}$). ∎

Proposition 1.16.
(a) There is an involutive automorphism ω of $\mathbb{Z}[X_n]^{S_n}$ defined by

$$\begin{array}{cccc} \omega\colon & \mathbb{Z}[X_n]^{S_n} & \longrightarrow & \mathbb{Z}[X_n]^{S_n} \\ & e_k & \longmapsto & h_k \end{array}$$

(b) $\omega(q_r(X_n;q,t)) = q_r(X_n;-t,-q)$ and $\omega(p_k) = (-1)^{k-1}p_k$.

Proof. The map ω is a well defined ring homomorphism since $\mathbb{Z}[X_n]^{S_n} = \mathbb{Z}[e_1, \ldots, e_n]$ is a polynomial ring. Comparing coefficients of z^k on each side of

$$1 = \left(\prod_{i=1}^n (1 - x_i z)\right)\left(\prod_{i=1}^n \frac{1}{1 - x_i z}\right) \qquad \text{yields} \qquad 0 = \sum_{r=1}^k (-1)^r e_r h_{n-r}.$$

Thus $e_1 = h_1$, and

$$h_k = \sum_{i=1}^k (-1) e_i h_{k-i} \qquad \text{and} \qquad e_k = (-1)^{-k} \sum_{i=0}^k (-1) e_i h_{k-i} = \sum_{j=1}^k (-1)^{-j-1} e_{k-j} h_j. \tag{1.17}$$

From the first of these relations, by induction on k,

$$\omega(h_k) = \sum_{i=1}^k (-1)^{i+1} h_i e_{k-i},$$

and, by comparing this identity with the second relation in (???) shows that $\omega(h_k) = e_k$. Hence $\omega^2 = \mathrm{id}$.

(b) ???? ∎

For a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$ of k define

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots \qquad \text{so that} \qquad \frac{n!}{z_\lambda} = \mathrm{Card}(\{w \in S_k \mid w \text{ has cycle type } \lambda\}) \tag{1.18}$$

is the size of the conjugacy class indexed by λ in the symmetric group S_k. Recalling that

$$\ln(1 - x_i y_j) = \sum_{k \ge 1} \frac{x_i^k y_j^k}{k} \quad \text{since} \quad \ln(1-t) = \int \frac{1}{1-t} dt = \int (1 + t + t^2 + \cdots) dt,$$

we have

$$\begin{aligned}\prod_{i,j}\frac{1}{1-x_iy_j} &= \exp\ln\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right) = \exp\left(\sum_{i,j}\ln(1-x_iy_j)\right) = \exp\left(\sum_k\sum_{i,j}\frac{x_i^ky_j^k}{k}\right)\\ &= \exp\left(\sum_k\frac{p_k(x)p_k(y)}{k}\right) = \prod_k\exp\left(\frac{p_k(x)p_k(y)}{k}\right) = \prod_k\sum_{m_k\geq 0}\left(\frac{p_k^{m_k}(x)p_k^{m_k}(y)}{k^{m_k}m_k!}\right)\\ &= \sum_{m_1,m_2,\dots}\left(\frac{p_1^{m_1}(x)p_2^{m_2}(x)\cdots p_1^{m_1}(y)p_2^{m_2}(y)\cdots}{1^{m_1}m_1!2^{m_2}m_2!\cdots}\right) = \sum_\lambda\frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}\end{aligned}$$

Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.