Numbers

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Calculus is the study of

(1) Derivatives
(2) Integrals
(3) Applications of derivatives
(4) Applications of integrals

A *derivative* is a creature you put a function into, it chews on it, and spits out a new function.

A *function* takes in a number, chews on it, and spits out a new number.

Derivatives	**Functions**
$\begin{matrix}\text{input}\\ \text{function}\end{matrix} \longrightarrow \dfrac{d}{dx} \longrightarrow \begin{matrix}\text{output}\\ \text{function}\end{matrix}$	$\begin{matrix}\text{input}\\ \text{number}\end{matrix} \longrightarrow f \longrightarrow \begin{matrix}\text{output}\\ \text{number}\end{matrix}$

The *integral* is the derivative backwards:

Numbers are at the bottom of the food chain.

At some point humankind wanted to count things and discovered the **positive integers**,

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for counting something,
BUT what if you don't have anything? How do we talk about nothing, nulla, zilch?
... and so we discovered the **nonnegative integers**,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for adding,

$$5+3=8,\quad 0+10=10,\quad 21+37=48,$$

BUT not so great for subtraction,

$$5-3=2,\quad 2-0=2,\quad 12-34=???.$$

... and so we discovered the **integers**

$$\ldots, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\ .$$

GREAT for adding, subtracting and multiplying,

$$3 \cdot 6 = 18, \quad -3 \cdot 2 = -6, \quad 0 \cdot 7 = 0,$$

BUT not so great if you only want part of the sausage ...,
... and so we discovered the **rational numbers**,

$$\frac{a}{b}\,, \qquad a \text{ an integer, } b \text{ an integer, } b \neq 0.$$

GREAT for addition, subtraction, multiplication, and division,
BUT not so great for finding $\sqrt{2} =$????,
... and so we discovered the **real numbers**,

all decimal expansions.

Examples:

$$\begin{aligned} \pi &= 3.1415926\ldots\ , \\ e &= 2.71828\ldots\ , \\ \sqrt{2} &= 1.414\ldots\ , \\ 10 &= 10.0000\ldots\ , \end{aligned} \qquad \begin{aligned} \frac{1}{3} &= .3333\ldots\ , \\ \frac{1}{8} &= .125 = .125000000\ldots\ , \end{aligned}$$

GREAT for addition, subtraction, multiplication, and division,
BUT not so great for finding $\sqrt{-9} =$????,
... and so we discovered the **complex numbers**,

$$a + bi, \qquad a \text{ a real number, } b \text{ a real number, } i = \sqrt{-1}\}.$$

Examples: $\quad 3 + \sqrt{2}i, \qquad 6 = 6 + 0i, \qquad \pi + \sqrt{7}i,$

and

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i.$$

GREAT.
Addition: $\quad (3 + 4i) + (7 + 9i) = 3 + 7 + 4i + 9i = 10 + 13i.$
Subtraction: $\quad (3 + 4i) - (7 + 9i) = 3 - 7 + 4i - 9i = -4 - 5i.$
Multiplication:

$$\begin{aligned} (3 + 4i)(7 + 9i) &= 3(7 + 9i) + 4i(7 + 9i) \\ &= 21 + 27i + 28i + 36i^2 \\ &= 21 + 55i - 36 \\ &= -15 + 55i. \end{aligned}$$

Division:

$$\frac{3+4i}{7+9i} = \frac{(3+4i)}{(7+9i)}\frac{(7-9i)}{(7-9i)} = \frac{21-27i+28i+36}{49-63i+63i+81}$$

$$= \frac{57+i}{130} = \frac{57}{130} + \frac{1}{130}i.$$

Square Roots: We want $\sqrt{-3+4i}$ to be some $a+bi$.

$$\text{If} \qquad \sqrt{-3+4i} = a+bi$$

then

$$\begin{aligned} -3+4i = (a+bi)^2 &= a^2+abi+abi+b^2i^2 \\ &= a^2-b^2+2abi. \end{aligned}$$

So

$$a^2-b^2=-3 \qquad \text{and} \qquad 2ab=4.$$

Solve for a and b.

$$b = \frac{4}{2a} = \frac{2}{a}. \qquad \begin{aligned} &\text{So} \quad a^2 - \left(\frac{2}{a}\right)^2 = -3. \\ &\text{So} \quad a^2 - \frac{4}{a^2} = -3. \\ &\text{So} \quad a^4 - 4 = -3a^2. \\ &\text{So} \quad a^4 + 3a^2 - 4 = 0. \\ &\text{So} \quad (a^2+4)(a^2-1) = 0. \end{aligned}$$

So $a^2 = -4$ or $a^2 = 1$.
So $a = \pm 1$, and $b = \frac{2}{\pm 1} = 2$ or -2.
So $a+bi = 1+2i$ or $a+bi = -1-2i$.
So $\sqrt{-3+4i} = \pm(1+2i)$.

Graphing:

Factoring:

$$x^2+5 = (x+\sqrt{5}\,i)(x-\sqrt{5}\,i),$$
$$x^2+x+1 = \left(x - \left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)$$

This is REALLY why we like the complex numbers. The **fundamental theorem of algebra** says that ANY POLYNOMIAL (for example, $x^{12673}+2563x^{159}+\pi x^{121}+\sqrt{7}\,x^{23}+9621\frac{1}{2}$) can be factored completely as

$$(x-u_1)(x-u_2)\cdots(x-u_n)$$

where $u_1,\ldots,u_n$ are complex numbers.