Symmetric functions: Definition of the Schur functions
Lecture Notes

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1. The $GL_n(\mathbb{C})$ case

The following fundamental example motivates the general case.

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the $\mathbb{Z}$-basis of $\mathbb{Z}^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z}\}$ given by $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the ith entry, so that

$$\begin{aligned} &P = \mathbb{Z}^n = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\}, \\ \text{and let} \qquad &P^+ = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}, \\ \text{and} \qquad &P^{++} = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in \mathbb{Z}^n \mid \lambda_1 > \cdots > \lambda_n\}. \end{aligned} \tag{1.1}$$

Then P^+ is a set of representatives of the orbits of the action of the symmetric group S_n on $\mathbb{Z}^n$ given by permuting the coordinates,

$$w\varepsilon_i = \varepsilon_{w(i)}, \qquad \text{for } w \in S_n,\ 1 \leq i \leq n. \tag{1.2}$$

There is a bijection

$$\begin{matrix} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{matrix} \qquad \text{where} \quad \rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}. \tag{1.3}$$

The group algebra of P is

$$\mathbb{Z}[P] = \mathbb{Z}\text{-span}\{x^\lambda \mid \lambda \in \mathbb{Z}^n\} \qquad \text{with} \quad x^\lambda x^\mu = x^{\lambda+\mu}, \qquad \text{for } \lambda, \mu \in \mathbb{Z}^n. \tag{1.4}$$

For $1 \leq i \leq n$ write

$$x_i = x^{\varepsilon_i} \qquad \text{so that} \qquad x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \quad \text{for } \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n,$$

and $\mathbb{Z}[P] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The action of S_n on $\mathbb{Z}^n$ induces an action of S_n on $\mathbb{Z}[P]$ given by

$$wx^\lambda = x^{w\lambda}, \qquad \text{for } w \in S_n,\ \lambda \in \mathbb{Z}^n. \tag{1.5}$$

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so that

$$wx_i = x_{w(i)}, \qquad \text{for } w \in S_n \text{ and } 1 \le i \le n, \tag{1.6}$$

The ring of *symmetric functions* is

$$\begin{aligned} \mathbb{Z}[P]^{S_n} &= \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} \\ &= \{f \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid f(x_{w(1)}, \ldots, x_{w(n)}) = f(x_1, \ldots, x_n) \text{ for all } w \in S_n\}, \end{aligned} \tag{1.7}$$

The *orbit sums*, or *monomial symmetric functions*, are

$$m_\lambda = \sum_{\gamma \in S_n\lambda} x^\gamma, \qquad \text{for } \lambda \in P^+,$$

where $S_n\lambda$ is the orbit of λ under the action of S_n. Then

$$\{m_\lambda \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[X]^{S_n}. \tag{1.8}$$

The set of *skew polynomials* is

$$\mathbb{Z}[P]^\epsilon = \{g \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid wg = \det(w)g \text{ for all } w \in S_n\}.$$

If $f \in \mathbb{Z}[P]_n^S$ and $g \in \mathbb{Z}[P]^\epsilon$ then $fg \in \mathbb{Z}[P]^\epsilon$ and so $\mathbb{Z}[P]^\epsilon$ is a $\mathbb{Z}[X_n]^{S_n}$-module. Let ϵ be the element of the group algebra of S_n given by

$$\epsilon = \sum_{w \in S_n} \det(w)w, \tag{1.9}$$

and define

$$a_\mu = \epsilon(x^\mu) = \sum_{w \in S_n} \det(w)wx^\mu, \qquad \text{for } \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\ge 0}. \tag{1.10}$$

Then

$$a_\mu = \det(w)a_{w\mu} \qquad \text{and} \qquad a_\mu = 0, \text{ if } \mu_i = \mu_j \text{ for some } i \ne j. \tag{1.11}$$

Using that $\{x^\lambda \mid \lambda \in P\}$ is a basis of $\mathbb{Z}[P]$ it follows that

$$\{a_\mu \mid \mu \in P^+ +\} = \{a_{\lambda+\rho} \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[P]^\epsilon. \tag{1.12}$$

Thus

$$\mathbb{Z}[P]^\epsilon = \epsilon \cdot \mathbb{Z}[P]. \tag{1.13}$$

The polynomial $x_j - x_i$ divides $a_{\lambda+\rho}$ since setting $x_i = x_j$ in the determinantal expression for $a_{\lambda+\rho}$ makes it equal to 0, and thus

$$a_{\lambda+\rho} = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ & \vdots & \cdots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n} \end{pmatrix} \quad \text{is divisible by} \quad \prod_{n \ge j > i \ge 1} (x_j - x_i). \tag{1.14}$$

since the polynomials $x_j - x_i$, $1 \le i < j \le n$ are coprime in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. When $\lambda = 0$, comparing coefficients of the maximal terms in $a_{\lambda+\rho}$ and $\prod(x_j - x_i)$ shows that the *Vandermonde determinant*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^0 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^0 \\ \vdots & & \cdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^0 \end{pmatrix} = \prod_{n \ge j > i \ge 1} (x_j - x_i). \tag{1.15}$$

Since $\{a_{\lambda+\rho} \mid \lambda \in P^+\}$ is a basis of A_n, and each $a_{\lambda+\rho}$ is divisible by a_ρ, the inverse of the map

$$\begin{array}{ccc} \mathbb{Z}[x_1, \ldots, x_n]^{S_n} & \longrightarrow & A_n \\ f & \longmapsto & a_\rho f \end{array} \tag{1.16}$$

is well defined, and thus it is an isomorphism of $\mathbb{Z}[P]^{S_n}$-modules.

The *Schur functions* are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P.$$

Since $\{a_{\lambda+\rho} \mid \lambda \in P^+\}$ is a basis of A_n and the map in (???) is an isomorphism,

$$\{s_\lambda \mid \lambda \in P^+\} \qquad \text{is a basis of } \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

The *dot action* of S_n on $\mathbb{Z}^n$ is given by

$$w \circ \mu = w(\mu + \rho) - \rho, \qquad \text{for } w \in S_n,\ \mu \in \mathbb{Z}^n. \tag{1.17}$$

The first relation in ??? and the definition of s_μ imply that

$$s_{w \circ \mu} = \det(w) s_\mu, \qquad \text{for } \mu \in P,\ w \in S_n. \tag{1.18}$$

2. The general case

A *lattice* is a free $\mathbb{Z}$-module. Let P be a lattice with a ($\mathbb{Z}$-linear) action of a finite group W so that P is a module for the group algebra $\mathbb{Z}W$. Extending coefficients, define

$$\mathfrak{h}_\mathbb{R}^* = \mathbb{R} \otimes_\mathbb{Z} P \qquad \text{and} \qquad \mathfrak{h}^* = \mathbb{C} \otimes_\mathbb{R} \mathfrak{h}_\mathbb{R}^*,$$

so that $\mathfrak{h}_\mathbb{R}^*$ and $\mathfrak{h}^*$ are vector spaces which are modules for the group algebras $\mathbb{R}W$ and $\mathbb{C}W$, respectively.

Assume that the action of W on $\mathfrak{h}_\mathbb{R}^*$ has fundamental regions???, and fix a fundamental region C in $\mathfrak{h}_\mathbb{R}^*$. Define

$$P^+ = P \cap \overline{C} \qquad \text{and} \qquad P^{++} = P \cap C$$

so that P^+ is a set of representatives of the orbits of the action of W on P. Assume???? that P^+ is a cone in P (a module for the monoid $\mathbb{Z}_{\ge 0}$). A set of *fundamental weights* is a set of $\omega_1, \ldots, \omega_n$ generators of (the $\mathbb{Z}_{\ge 0}$-module) P^+ which also form a $\mathbb{Z}$-basis of P. There is a bijection

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{where} \quad \rho = \omega_1 + \ldots + \omega_n. \tag{2.1}$$

Let $\langle,\rangle\colon \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ be a W-invariant symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$ (such that the restriction to P is a perfect pairing??? with values in $\mathbb{Z}$???). The *simple coroots* are $\alpha_1^\vee, \ldots, \alpha_n^\vee$ the dual basis to the fundamental weights,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{2.2}$$

Define

$$\overline{C^\vee} = \sum_{i=1}^n \mathbb{R}_{\leq 0}\alpha_i^\vee \qquad \text{and} \qquad C^\vee = \sum_{i=1}^n \mathbb{R}_{<0}\alpha_i^\vee. \tag{2.3}$$

The *dominance order* is the partial order on $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\lambda \geq \mu \qquad \text{if} \qquad \mu \in \lambda + \overline{C^\vee}. \tag{2.4}$$

The group algebra of the abelian group P is

$$\mathbb{Z}[P] = \mathbb{Z}\text{-span}\{x^\lambda \mid \lambda \in P\} \qquad \text{with} \quad x^\lambda x^\mu = x^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in P. \tag{2.5}$$

The action of W on P induces an action of W on $\mathbb{Z}[P]$ given by

$$wx^\lambda = x^{w\lambda}, \qquad \text{for } w \in W,\ \lambda \in P. \tag{2.6}$$

The ring of *symmetric functions* is

$$\mathbb{Z}[P]^W = \{f \in \mathbb{Z}[P] \mid wf = f \text{ for all } w \in W\}, \tag{2.7}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in W\lambda} x^\gamma, \qquad \text{for } \lambda \in P^+,$$

where $W\lambda$ is the orbit of λ under the action of W. Then

$$\{m_\lambda \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[P]^W. \tag{2.8}$$

Theorem 2.9. *$\mathbb{Z}[P]$ is a free $\mathbb{Z}[P]^W$ of rank $|W|$.*

Proof. ?????? ∎

The set of *skew polynomials* is

$$\mathbb{Z}[P]^\epsilon = \{g \in \mathbb{Z}[P] \mid wg = \det(w)g \text{ for all } w \in W\}.$$

If $f \in \mathbb{Z}[X_n]_n^S$ and $g \in \mathbb{Z}[P]^\epsilon$ then $fg \in \mathbb{Z}[P]^\epsilon$ and so A_n is a $\mathbb{Z}[P]^W$-module. Let ϵ be the element of the group algebra of W given by

$$\epsilon = \sum_{w \in W} \det(w)w, \tag{2.10}$$

and define

$$a_\mu = \epsilon(x^\mu) = \sum_{w \in W} \det(w)wx^\mu, \qquad \text{for } \mu \in P. \tag{2.11}$$

Then

$$a_\mu = \det(w) a_{w\mu} \qquad \text{and} \qquad a_\mu = 0, \text{ if } \langle \mu, \alpha^\vee \rangle = 0 \text{ for some } \alpha \in R^+. \tag{2.12}$$

Using that $\{x^\lambda \mid \lambda \in P\}$ is a basis of $\mathbb{Z}[P]$ it follows that

$$\{a_\mu \mid \mu \in P^{++}\} = \{a_{\lambda+\rho} \mid \lambda \in P^+\} \qquad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[P]^\epsilon. \tag{2.13}$$

Thus

$$\mathbb{Z}[P]^\epsilon = \epsilon \cdot \mathbb{Z}[P]. \tag{2.14}$$

Let $f \in \mathbb{Z}[P]^\epsilon$ and let $\alpha \in R^+$. If f_γ is the coefficient of x^γ in f then

$$\sum_{\gamma \in P} f_\gamma x^\gamma = f = -s_\alpha f = \sum_{\gamma \in P} -f_\gamma x^{s_\alpha \gamma}, \qquad \text{and so} \qquad f = \sum_{\substack{\gamma \in P \\ \langle \gamma, \alpha^\vee \rangle \geq 0}} f_\gamma (x^\gamma - x^{s_\alpha \gamma}),$$

since $f_{s_\alpha \gamma} = -f_\gamma$. Since each term $x^\gamma - x^{s_\alpha \gamma}$ is divisible $1 - x^{-\alpha}$, f is divisible by $1 - x^{-\alpha}$, and thus

$$\text{each} \quad f \in \mathbb{Z}[P]^\epsilon \quad \text{is divisible by} \quad x^\rho \prod_{\alpha \in R^+} (1 - x^{-\alpha}) = \prod_{\alpha \in R^+} (x^{\alpha/2} - x^{-\alpha/2}). \tag{2.15}$$

since the polynomials $1 - x^{-\alpha}$, $\alpha \in R^+$ are coprime in $\mathbb{Z}[P]$ (and x^ρ is a unit in $\mathbb{Z}[P]$). Comparing coefficients of the maximal terms in a_ρ and $x^\rho \prod_{\alpha \in R^+} (1 - x^{-\alpha})$ shows that the *Weyl denominator*,

$$a_\rho = \prod_{\alpha \in R^+} (x^{\alpha/2} - x^{-\alpha/2}) = x^\rho \prod_{\alpha \in R^+} (1 - x^{-\alpha}). \tag{2.16}$$

Since each $f \in \mathbb{Z}[P]^\epsilon$ is divisible by a_ρ the inverse of the map

$$\begin{array}{ccc} \mathbb{Z}[P]^W & \longrightarrow & \mathbb{Z}[P]^\epsilon \\ f & \longmapsto & a_\rho f \end{array} \tag{2.17}$$

is well defined and, thus, is an isomorphism of $\mathbb{Z}[P]^W$-modules.

The *Schur functions* or *Weyl characters* are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P. \tag{2.18}$$

Since $\{a_{\lambda+\rho} \mid \lambda \in P^+\}$ is a basis of $\mathbb{Z}[P]^\epsilon$ and the map in (???) is an isomorphism,

$$\{s_\lambda \mid \lambda \in P^+\} \qquad \text{is a basis of } \mathbb{Z}[P]^W. \tag{2.19}$$

The *dot action* of S_n on P is given by

$$w \circ \mu = w(\mu + \rho) - \rho, \qquad \text{for } w \in S_n, \, \mu \in P. \tag{2.20}$$

The first relation in ??? and the definition of s_μ imply that

$$s_{w \circ \mu} = \det(w) s_\mu, \qquad \text{for } \mu \in P, \, w \in W. \tag{2.21}$$

Lemma 2.22. *Let $f \in \mathbb{Z}[P]^W$ and write $f = \sum_\gamma f_\gamma x^\gamma$ so that f_γ is the coefficient of x^γ in f. Then*

$$f = \sum_{\mu \in P^+} f_\mu m_\mu = \sum_{\lambda \in P^+} \eta^\lambda s_\lambda, \qquad \text{where} \qquad \eta^\lambda = \sum_{w \in W} \det(w) f_{\lambda+\rho-w\rho}.$$

Proof. The first equality is immediate from the definition of m_μ. Since $f \in \mathbb{Z}[P]^W$, $f\epsilon(x^\rho) = \epsilon(fx^\rho)$ and $f_\mu = f_{w^{-1}\mu}$, and so

$$\begin{aligned} f &= \frac{1}{a_\rho} f a_\rho = \frac{1}{a_\rho} f\epsilon(x^\rho) = \frac{1}{a_\rho}\epsilon(fx^\rho) = \sum_{\gamma \in P} f_\gamma \frac{\epsilon(x^{\gamma+\rho})}{a_\rho} \\ &= \sum_{\gamma \in P} f_\gamma s_\gamma = \sum_{\lambda \in P^+} \sum_{w \in W} f_{w \circ \lambda} s_{w \circ \lambda} = \sum_{\lambda \in P^+} s_\lambda \sum_{w \in W} \det(w) f_{w \circ \lambda} \\ &= \sum_{\lambda \in P^+} s_\lambda \sum_{w \in W} \det(w) f_{w^{-1}(w \circ \lambda)} = \sum_{\lambda \in P^+} s_\lambda \sum_{w \in W} \det(w)^{-1} f_{\lambda+\rho-w\rho}, \end{aligned}$$

which establishes the second equality. ∎

Define positive integers $p(\gamma)$ by

$$\prod_{\alpha \in R^+} \frac{1}{1-x^{-\alpha}} = \sum_{\gamma \in Q^+} p(\gamma) x^{-\gamma}. \tag{2.23}$$

Corollary 2.24. *Let $K_{\lambda\mu}$ be the integers defined by*

$$s_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu} m_\mu, \qquad \text{for } \lambda \in P^+. \tag{2.25}$$

Then $\quad K_{\lambda\lambda} = 1 \quad$ *and* $\quad K_{\lambda\mu} = 0$ *unless* $\mu \leq \lambda$, *and*

$$K_{\lambda\mu} = \sum_{w \in W} \det(w) p(w(\lambda+\rho) - (\mu + rho)).$$

Proof. If $w \neq 1$ then $w(\lambda+\rho) < \lambda+\rho$ so that $w(\lambda+\rho)-\rho < \lambda$ and

$$s_\lambda = \left(\sum_{w \in W} \det(w) e^{w(\lambda+\rho)-\rho}\right) \cdot \frac{1}{1-x^{-\alpha}} = x^\lambda + ((\text{lower terms in dominance order}).$$

Thus $K_{\lambda\lambda} = 1$ and $K_{\lambda\mu} = 0$ unless $\mu \leq \lambda$. The coefficient of x^μ in

$$s_\lambda = \left(\sum_{w \in W} \det(w) x^{w(\lambda+\rho)-\rho}\right) \prod_{\alpha \in R^+} \frac{1}{1-x^{-\alpha}} = \sum_{\substack{w \in W \\ \gamma \in Q^+}} \det(w) p(\gamma) x^{w(\lambda+\rho)-\gamma-\rho},$$

has a contribution only when $w(\lambda+\rho)-\gamma-\rho=\mu$ so that $\gamma=w(\lambda+\rho)-(\mu+\rho)$. Thus

$$K_{\lambda\mu}=\sum_{w\in W}\det(w)p(w(\lambda+\rho)-(\mu+rho)).$$

∎

Define integers $c^\lambda_{\mu\nu}$ by

$$s_\mu s_\nu=\sum_{\lambda\in P^+}c^\lambda_{\mu\nu}s_\lambda,\qquad \text{for } \mu,\nu\in P^+.$$

Then $c^\lambda_{\mu\nu}$ is the coefficient of s_λ in ????????

3. Other examples

Example. The $Sp_{2n}(\mathbb{C})$ case.

Let $W=WC_n$ be the group of $n\times n$ matrices with
(a) exactly one nonzero entry in each row and each column,
(b) the nonzero entries are ±1.

Then $W=WC_n=O_n(\mathbb{Z})$, the group of orthogonal matrices with entries in $\mathbb{Z}$. Let $\varepsilon_1,\dots,\varepsilon_n$ be the $\mathbb{Z}$-basis of $\mathbb{Z}^n=\{(\lambda_1,\dots,\lambda_n)\mid\lambda_i\in\mathbb{Z}\}$ given by $\varepsilon_i=(0,\dots,0,1,0,\dots,0)$, with the 1 in the ith entry, so that

$$\begin{aligned} &P=\mathbb{Z}^n=\mathbb{Z}\text{-span}\{\varepsilon_1,\dots,\varepsilon_n\},\\ \text{and let}\quad &P^+=\{\lambda=\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n\in\mathbb{Z}^n\mid\lambda_1\ge\cdots\ge\lambda_n\ge0\},\\ \text{and}\quad &P^{++}=\{\lambda=\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n\in\mathbb{Z}^n\mid\lambda_1>\cdots>\lambda_n>0\}.\end{aligned}\tag{3.1}$$

Then P^+ is a set of representatives of the orbits of the action of the natural action of W on P. There is a bijection

$$\begin{matrix}P^+&\longrightarrow&P^{++}\\ \lambda&\longmapsto&\rho+\lambda\end{matrix}\qquad\text{where}\quad\rho=n\varepsilon_1+(n-1)\varepsilon_2+\cdots+2\varepsilon_{n-1}+\varepsilon_n.\tag{3.2}$$

Let

$$\mathbb{Z}[P]=\mathbb{Z}\text{-span}\{x^\lambda\mid\lambda\in P\}\qquad\text{with}\quad x^\lambda x^\mu=x^{\lambda+\mu},\qquad\text{for }\lambda,\mu\in P.\tag{3.3}$$

For $1\le i\le n$ write

$$x_i=x^{\varepsilon_i}\qquad\text{so that}\qquad x^\lambda=x_1^{\lambda_1}\cdots x_n^{\lambda_n}\quad\text{for }\lambda=\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n,$$

and $\mathbb{Z}[P]=\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1}]$.

Example. The $\mathrm{Spin}_{2n+1}(\mathbb{C})$ case.

Let $W=WB_n=WC_n$, where $WC_n=O_n(\mathbb{Z})$. Let

$$\begin{aligned}&\omega_1=\tfrac12(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n),\\ &\omega_i=\varepsilon_i+\varepsilon_{i+1}+\cdots+\varepsilon_n,\qquad\text{for }2\le i\le n,\end{aligned}$$

so that

$$
\begin{aligned}
P &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \text{all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \tfrac{1}{2} + \mathbb{Z}\}, \\
P^+ &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P \mid 0 \le \lambda_1 \le \cdots \le \lambda_n\}, \\
P^{++} &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P \mid 0 < \lambda_1 < \cdots < \lambda_n\}, \\
\rho &= \varepsilon_1 + 2\varepsilon_2 + \cdots + n\varepsilon_n - \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n).
\end{aligned}
\tag{3.4}
$$

Example. The $\mathrm{Spin}_{2n}(\mathbb{C})$ case.

Let $W = WD_n$ be the group of $n \times n$ matrices with

(a) exactly one nonzero entry in each row and each column,
(b) the nonzero entries are ± 1, and
(c) there are an even number of -1 entries.

Then WD_n is a normal subgroup of index 2 in $WC_n = O_n(\mathbb{Z})$. Let

$$
\begin{aligned}
\omega_1 &= \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n), \\
\omega_2 &= \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n), \\
\omega_i &= \epsilon_i + \epsilon_{i+1} + \ldots + \epsilon_n, \qquad \text{for } i > 2,
\end{aligned}
$$

so that

$$
\begin{aligned}
P &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_1 \in \tfrac{1}{2}\mathbb{Z} \text{ and, for } i > 1, \text{ all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \tfrac{1}{2} + \mathbb{Z}\}, \\
P^+ &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P \mid |\lambda_1| \le \lambda_2 \le \cdots \le \lambda_n\}, \\
P^{++} &= \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in P \mid |\lambda_1| < \cdots < \lambda_n\}, \\
\rho &= \varepsilon_1 + 2\varepsilon_2 + \cdots + n\varepsilon_n - (\varepsilon_1 + \cdots + \varepsilon_n).
\end{aligned}
\tag{3.5}
$$

Example. The $SL_n(\mathbb{C})$-case

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the $\mathbb{R}$-basis of $\mathbb{R}^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{R}\}$ given by $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the ith entry. The symmetric group S_n acts on $\mathbb{R}^n$ by permuting the coordinates and, by restriction, S_n acts on

$$\mathfrak{h}^*_{\mathbb{R}} = \{\gamma = \gamma_1\varepsilon_1 + \cdots + \gamma_n\varepsilon_n \mid \gamma_i \in \mathbb{R}, \gamma_1 + \cdots + \gamma_n = 0\}.$$

Let

$$\omega_n = \varepsilon_1 + \cdots + \varepsilon_n.$$

Then S_n acts also on the $\mathbb{Z}$-submodule of $\mathfrak{h}^*_{\mathbb{R}}$ given by

$$P = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n - \tfrac{|\lambda|}{n}\omega_n \mid \lambda_i \in \mathbb{Z}_{\ge 0}\},$$

which has $\mathbb{Z}$-basis $\{\omega_1, \ldots, \omega_{n-1}\}$ where

$$\omega_i = \varepsilon_1 + \cdots + \omega_i - \frac{1}{n}(\omega_n), \qquad \text{for } 1 \le i \le n-1. \tag{3.6}$$

Then

$$\begin{aligned} P^+ &= \{\lambda \in P \mid \lambda_1 \geq \cdots \geq \lambda_n\}, \\ P^{++} &= \{\lambda \in P \mid \lambda_1 > \cdots > \lambda_n\}, \\ \rho &= (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} - \left(\tfrac{n-1}{2}\right)\omega_n. \end{aligned} \tag{3.7}$$

Let

$$\mathbb{Z}[P] = \mathbb{Z}\text{-span}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}, \quad \text{for } \lambda, \mu \in P. \tag{3.8}$$

For $1 \leq i \leq n$ write

$$x_i = X^{\varepsilon_i - \frac{1}{n}\omega_n} \qquad \text{so that} \qquad X^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \quad \text{for } \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n - \tfrac{|\lambda|}{n}\omega_n \in P.$$

Then $\mathbb{Z}[P]$ is the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by the ideal generated by the element $x_1 \cdots x_n - 1$,

$$\mathbb{Z}[P] = \frac{\mathbb{Z}[x_1^{\pm1}, \ldots, x_n^{\pm1}]}{\langle x_1 \cdots x_n - 1\rangle}.$$

The action of S_n on P induces an action of S_n on $\mathbb{Z}[P]$ given by

$$wx_i = x_{w(i)}, \qquad \text{for } w \in S_n \text{ and } 1 \leq i \leq n, \tag{3.9}$$

and the ring of *symmetric functions* is

$$\mathbb{Z}[P]^{S_n} = \{f \in \mathbb{Z}[P] \mid wf = f \text{ for all } w \in S_n\}, \tag{3.10}$$

Example: Type A_2.

Example: Type B_2.

Example: Type C_2,

Example: Type G_2.

Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.