An approach to "early trascendentals"

Arun Ram
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Version: May 20, 2004

The function god(t)

There is one function that

(a) in the Beginning, created something from nothing, and

(b) is "unchanging", or rather, its change is itself.

Through the ages thinkers have contemplated this function and nowadays it is common to write (a) and (b) in abbreviated form,

$$(\text{a}') \quad \text{god}(0) = 1, \qquad \text{and} \qquad (\text{b}') \quad \frac{d\ \text{god}(t)}{dt} = \text{god}(t),$$

but the meaning is still the same.

Two of the children of god are eve and adam:

$$\text{god}(it) = \text{adam}(t) + i\ \text{eve}(t).$$

Trying to understand god(t)

If we try to "understand" god in "normal" terms,

$$\text{god}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots,$$

then

$$\begin{array}{llll} \text{since} & \text{god}(0) = 1, & a_0 = 1, & \text{and} \\ & & & \\ \text{since} & \dfrac{d\text{god}(t)}{dt} = \text{god}(t), & a_1 = a_0, & \text{and} \\ & & 2a_2 = a_1, & \text{and} \\ & & 3a_3 = a_2, & \text{and} \\ & & 4a_4 = a_3, & \text{and} \\ & & 5a_5 = a_4, & \ldots, \text{etc.,} \end{array}$$

Keywords: exponential functions, trigonometric functions

and so

$$\mathrm{god}(t) = 1 \ + \ t \ + \ \frac{1}{2!}t^2 \ + \ \frac{1}{3!}t^3 \ + \ \frac{1}{4!}t^4 \ + \ \cdots,$$

which illustrates that $\mathrm{god}(t)$ exists everywhere and goes on forever.

An amazing thing about $\mathrm{god}(t)$

One of the amazing things about god is that

$$\mathrm{god}(t+s) = \mathrm{god}(t)\ \mathrm{god}(s).$$

To see why god is this way suppose that there is a "different" function such that

(a″) is "unchanging" $\left(\text{i.e. } \dfrac{d\,\widetilde{\mathrm{god}}(t)}{dt} = \widetilde{\mathrm{god}}(t)\right)$, and

(b″) in the Beginning, was the way that god is after s millenia (i.e. $\widetilde{\mathrm{god}}(0) = \mathrm{god}(s)$).

By the chain rule,

$$\frac{d\ \mathrm{god}(t+s)}{dt} = \mathrm{god}(t+s) \qquad \text{and} \qquad \mathrm{god}(0+s) = \mathrm{god}(s),$$

and so

$$\mathrm{god}(t+s) = \widetilde{\mathrm{god}}(t).$$

Also,

$$\frac{d\ (\mathrm{god}(t)\mathrm{god}(s))}{dt} = \mathrm{god}(t)\mathrm{god}(s), \qquad \text{and} \qquad \mathrm{god}(0)\mathrm{god}(s) = \mathrm{god}(s),$$

and so

$$\mathrm{god}(t)\mathrm{god}(s) = \widetilde{\mathrm{god}}(t) = \mathrm{god}(t+s).$$

What about $\mathrm{adam}(t)$ **and** $\mathrm{eve}(t)$**?**

$$\begin{aligned}
\mathrm{god}(it) &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots \\
&= 1 + \frac{i^2t^2}{2!} + \frac{i^4t^4}{4!} + \frac{i^6t^6}{6!} + \cdots \\
&\qquad + it + \frac{i^3t^3}{3!} + \frac{i^5t^5}{5!} + \frac{i^7t^7}{7!} + \cdots \\
&= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\
&\qquad + it - \frac{it^3}{3!} + \frac{it^5}{5!} - \frac{it^7}{7!} + \cdots \\
&= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)
\end{aligned}$$

and, since adam and eve are the children of god,

$$\text{i.e.} \qquad \text{because} \qquad \mathrm{god}(it) = \mathrm{adam}(t) + i\ \mathrm{eve}(t)\ ,$$

we see that

$$\mathrm{adam}(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots, \qquad \text{and}$$

$$\mathrm{eve}(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} + \cdots,$$

from which it follows that

$$\mathrm{adam}(0) = 0, \qquad \mathrm{eve}(0) = 1,$$

$$\mathrm{adam}(-t) = -\mathrm{adam}(t), \qquad \mathrm{eve}(-t) = \mathrm{eve}(t),$$

$$\frac{d\ \mathrm{adam}(t)}{dt} = \mathrm{eve}(t), \qquad \frac{d\ \mathrm{eve}(t)}{dt} = -\mathrm{adam}(t).$$

So, adam and eve are complete opposites and identical twins at the same time.

Complete opposites and identical twins at the same time, another manifestation

$$\begin{aligned} 1 = \mathrm{god}(0) &= \mathrm{god}(it - it) = \mathrm{god}(it + i(-t)) = \mathrm{god}(it)\mathrm{god}(i(-t)) \\ &= (\mathrm{adam}(t) + i\ \mathrm{eve}(t))(\mathrm{adam}(-t) + i\ \mathrm{eve}(-t)) \\ &= (\mathrm{adam}(t) + i\ \mathrm{eve}(t))(\mathrm{adam}(t) - i\ \mathrm{eve}(t)) \\ &= (\mathrm{adam}(t))^2 + (\mathrm{eve}(t))^2, \end{aligned}$$

$$\text{i.e.} \qquad 1 = (\mathrm{adam}(t))^2 + (\mathrm{eve}(t))^2.$$

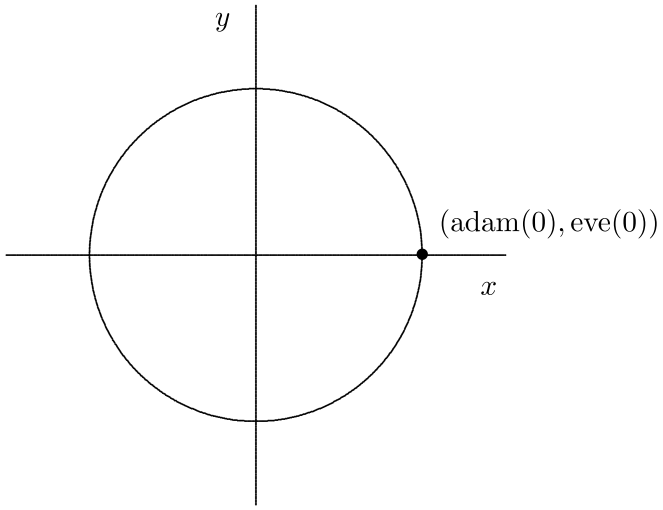
Through the ages: where are we now?

Let $x = \mathrm{eve}(t)$ and $y = \mathrm{adam}(t)$.

(A) In the Beginning the point (x, y) was at $(\mathrm{adam}(0), \mathrm{eve}(0)) = (1, 0)$, and

since $1 = \mathrm{adam}(t))^2 + (\mathrm{eve}(t))^2$, $\qquad x^2 + y^2 = 1$, $\qquad$ and

(B) adam and eve travel through the ages on a circle of radius 1.



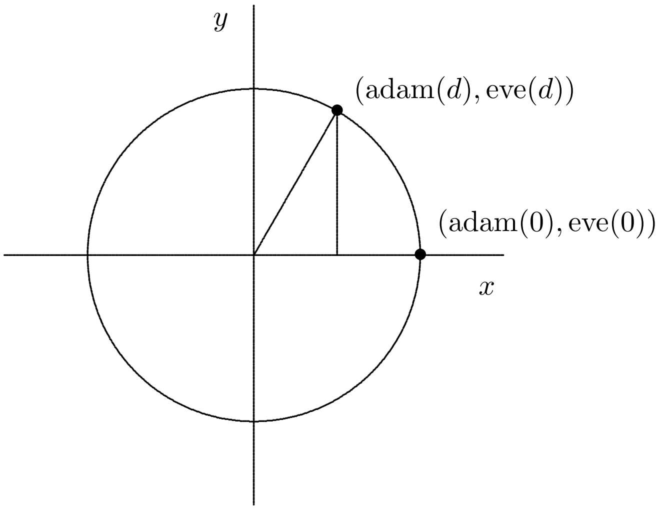
Where are they after d millenia?

$$\begin{aligned}
\begin{array}{c}\text{The distance traveled}\\ \text{after } d \text{ millenia}\end{array} &= \int_{t=0}^{t=d} ds = \int_{t=0}^{t=d} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_{t=0}^{t=d} \sqrt{\left(\frac{d\ \mathrm{adam}(t)}{dt}\right)^2 + \left(\frac{d\ \mathrm{eve}(t)}{dt}\right)^2}\, dt \\
&= \int_{t=0}^{t=d} \sqrt{(\mathrm{eve})^2 + (-\mathrm{adam}(t))^2}\, dt \\
&= \int_{t=0}^{t=d} \sqrt{1}\, dt = \int_{t=0}^{t=d} dt = \left. t\ \right|_{t=0}^{t=d} = d - 0 = d,
\end{aligned}$$

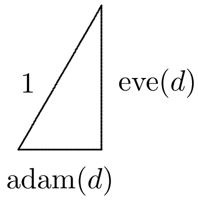
and so

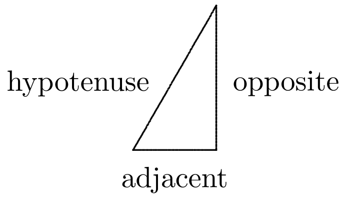
adam(t) = x-coordinate of the point on a circle of radius 1 which is distance d from the point (1,0), and

eve(t) = y-coordinate of the point on a circle of radius 1 which is distance d from the point (1,0).



The triangle in this picture is





and so

$$\mathrm{adam}(d) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{and} \qquad \mathrm{eve}(d) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

for a right triangle with angle d.

Some remarks on society

1. It is interesting to note that our school systems like to introduce our children to adam(t) and eve(t) but prefer to hide from my child how close they really are to god(t).

2. Mathematicians are a cloistered group and prefer to study god, adam, and eve in anonymity. In the mathematical literature

god(t)	is usually called	e^t ,	
adam(t)	is usually termed	$\cos t$,	and
eve(t)	is usually called	$\sin t$.	