The exponential function

Arun Ram
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

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Define the **exponential function** as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$

where k**-factorial** is

$$k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1, \quad \text{for } k = 1, 2, 3, \ldots.$$

Why would anyone be so crazy as to write down such a horrible mess??

Example: Is there a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

that changes addition into multiplication??,

$$f(x)f(y) = f(x+y).$$

If so

$$\begin{aligned} f(x+y) &= c_0 + c_1(x+y) + c_2(x+y)^2 + c_3(x+y)^3 + c_4(x+y)^4 + c_5(x+y)^5 + \cdots \\ &= c_0 \\ &\quad + c_1 x + c_1 y + \\ &\quad + c_2 x^2 + 2c_2 xy + c_2 y^2 + \\ &\quad + c_3 x^3 + 3c_3 x^2 y + 3c_3 x y^2 + c_3 y^3 + \\ &\quad + c_4 x^4 + 4c_4 x^3 y + 6c_4 x^2 y^2 + 4c_4 x y^3 + c_4 y^4 + \\ &\quad + \cdots \end{aligned}$$

must be equal to

$$\begin{aligned} f(x)f(y) &= (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4 + \cdots) \\ &= c_0^2 + c_0 c_1 x + c_0 c_2 x^2 + c_0 c_3 x^3 + c_0 c_4 x^4 + \cdots \\ &\qquad + c_0 c_1 y + c_1^2 xy + c_1 c_2 x^3 y + c_1 c_4 x^4 y + \cdots \\ &\qquad + \cdots. \end{aligned}$$

Comparing terms in these two expressions gives

$$\begin{array}{llllll} c_0^2 = c_0, & c_0c_1 = c_1, & c_0c_2 = c_2, & c_0c_3 = c_3, & c_0c_4 = c_4, & \ldots, \\ & c_0c_1 = c_1, & c_1^2 = 2c_2, & c_1c_2 = 3c_3, & c_1c_3 = 4c_4, & c_1c_4 = 5c_5, \quad \ldots, \end{array}$$

So

$$c_0 = 1, \quad c_2 = \frac{c_1^2}{2}, \quad c_1\frac{c_1^2}{2} = 3c_3, \quad c_1\frac{c_1^3}{3\cdot 2} = 4c_4, \quad c_1\frac{c_1^4}{4\cdot 3\cdot 2} = 5c_5, \quad \ldots.$$

So

$$c_0 = 1, \quad c_2 = \frac{c_1^2}{2}, \quad c_3 = \frac{c_1^3}{3\cdot 2\cdot 1}, \quad c_4 = \frac{c_1^4}{4\cdot 3\cdot 2\cdot 1}, \quad c_5 = \frac{c_1^5}{5\cdot 4\cdot 3\cdot 2\cdot 1}, \quad \ldots.$$

So

$$\begin{aligned} f(x) &= 1 + c_1x + \frac{c_1^2}{2}x^2 + \frac{c_1^3}{3!}x^3 + \frac{c_1^4}{4!}x^4 + \frac{c_1^5}{5!}x^5 + \cdots \\ &= 1 + c_1x + \frac{(c_1x)^2}{2} + \frac{(c_1x)^3}{3!} + \frac{(c_1x)^4}{4!} + \frac{(c_1x)^5}{5!} + \cdots \\ &= e^{c_1x}. \end{aligned}$$

So,

$$\text{if} \quad f(x+y) = f(x)f(y) \quad \text{then} \quad f(x) = e^{c_1x}.$$

Example: Is there a function

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$

whose derivative is itself,

$$\frac{df}{dx} = f \qquad ???$$

If so,

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$

must be equal to

$$\frac{df}{dx} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \cdots.$$

Comparing terms in these two expressions gives

$$c_1 = c_0, \quad 2c_2 = c_1, \quad 3c_3 = c_2, \quad 4c_4 = c_3, \quad 5c_5 = c_4, \quad 6c_6 = c_5, \quad \ldots$$

So

$$c_2 = \frac{c_0}{2}, \quad 3c_3 = \frac{c_0}{2}, \quad 4c_4 = \frac{c_0}{3\cdot 2}, \quad 5c_5 = \frac{c_0}{4\cdot 3\cdot 2}, \quad \ldots.$$

So

$$\begin{aligned} f(x) &= c_0 + c_0x + \frac{c_0}{2}x^2 + \frac{c_0^2}{3!}x^3 + \frac{c_0}{4!}x^4 + \frac{c_0}{5!}x^5 + \cdots \\ &= c_0(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots) \\ &= c_0e^x. \end{aligned}$$

So, if $\dfrac{df}{dx} = f$ then $f = c_0 e^x$.

So $f(x) = e^x$ is the ONLY function such that

$$e^{x+y} = e^x e^y \quad \text{and} \quad \frac{de^x}{dx} = e^x.$$

Example: Find e^0.

$$e^0 = 1 + 0 + \frac{0^2}{2!} + \frac{0^3}{3!} + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

Example: Explain why $e^{-x} = \dfrac{1}{e^x}$.

$$e^x e^{-x} = e^{x+(-x)} = e^{x-x} = e^0 = 1.$$

Divide both sides by e^x.

$$\text{So} \quad e^{-x} = \frac{1}{e^x}.$$

Define

$$(e^x)^y = e^{xy}.$$

The *exponential function* is the function e^x such that

$$\frac{de^x}{dx} = e^x \quad \text{and} \quad e^0 = 1.$$

Figure out what e^x is:

Suppose $e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
Then $e^0 = a_0 + 0 + 0 + \cdots = 1$. So $a_0 = 1$.

$$\frac{de^x}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$= e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

So $a_1 = 0$, $2a_2 = a_1$, $4a_3 = a_2$, $4a_4 = a_3$, ... So

$$a_0 = 1, \;\; a_1 = 1, \;\; a_2 = \frac{1}{2}, \;\; a_3 = \frac{1}{2\cdot 3}, \;\; a_4 = \frac{1}{2\cdot 3\cdot 4}, \;\; a_5 = \frac{1}{2\cdot 3\cdot 4\cdot 5}, \ldots$$

So

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4}x^4 + \frac{1}{2\cdot 3\cdot 4\cdot 5}x^5 + \cdots.$$

Factorials

$$7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$$
$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$$
$$3! = 3\cdot 2\cdot 1.$$

So

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

So

$$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots = 2.781828\ldots$$

$$e^{-3} = 1 + (-3) + \frac{(-3)^2}{2} + \frac{(-3)^3}{6} + \frac{(-3)^4}{24} + \cdots$$

$$= 1 - 3 + \frac{3^2}{2} - \frac{3^3}{6} + \frac{3^4}{24} + \cdots = ????$$

$$\underset{x}{\text{input}} \longrightarrow e^x \longrightarrow \underset{e^x}{\text{output}}$$

Note: By the chain rule

$$\frac{d}{dx}e^{2+x} = e^{2+x}\cdot\frac{d(2+x)}{dx} = e^{2+x}\cdot 1 = e^{2+x}$$

and

$$e^{2+0} = e^2.$$

So

$$e^{2+x} = e^2x + \frac{e^2x^2}{2!} + \frac{e^2x^3}{3!} + \cdots$$

since, in this case

$$a_0 = e^2, \ \ a_1 = a_0, \ \ 2a_2 = a_1, \ \ 3a_3 = a_2, \ldots$$

if

$$e^{2+x} = a_0 + a_1x + a_2x^2 + \cdots.$$

So

$$e^{2+x} = e^2e^x.$$

Similarly,

$$e^{10+x} = e^{10}e^x \ \text{ and } \ e^{642+x} = e^{542}e^x$$

and

$$e^{y+x} = e^ye^x \ .$$

Since $e^{-x}e^x = e^{-x+x} = e^0 = 1$

$$e^{-x} = \frac{1}{e^x}.$$

Since

$$\begin{aligned} e^{10x} &= e^{x+x+x+x+x+x+x+x+x+x} \\ &= e^xe^{x+x+x+x+x+x+x+x+x} \\ &= e^xe^xe^{x+x+x+x+x+x+x+x} \\ &= e^xe^xe^xe^{x+x+x+x+x+x+x} \\ &= e^xe^xe^xe^xe^{x+x+x+x+x+x} \\ &= e^xe^xe^xe^xe^xe^{x+x+x+x+x} \\ &= e^xe^xe^xe^xe^xe^xe^xe^xe^xe^x = (e^x)^{10} \end{aligned}$$

Summary: e^x is the function such that

$$\frac{de^x}{dx} = e^x \quad \text{and} \quad e^0 = 1.$$

Then

$$\begin{aligned} e^{x+y} &= e^xe^y \\ e^{-x} &= \frac{1}{e^x} \\ e^{nx} &= (e^x)^n. \end{aligned}$$