Derivatives

Arun Ram
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Version: May 20, 2004

A **function** eats a number, chews on it, and spits out another number.

PICTURE

A **constant function** always spits out the same number, no matter what the input is.

Example: $f(x) = 2$.

PICTURE

We call this function 2.
So, 2 *sometimes means the number* 2, and *sometimes means the function* 2.

A **derivative** eats a function, chews on it, and spits out another function.

PICTURE

The derivative $\frac{d}{dx}$ knows what to spit out by always following the rules:

(1) $\frac{dx}{dx} = 1$,

(2) $\frac{d(cf)}{dx} = c\frac{df}{dx}$, if c does not change when x changes,

(3) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$,

(4) $\frac{d(fg)}{dx} = f\,\frac{dg}{dx} + \frac{df}{dx}\,g$.

Example: Find $\frac{dy}{dx}$ if $y = 5x$.

$$\frac{dy}{dx} = \frac{d(5x)}{dx} = 5\,\frac{dx}{dx} = 5 \cdot 1 = 5.$$

Example: Find $\frac{dy}{dx}$ if $y = \pi x$.

$$\frac{dy}{dx} = \frac{d(\pi x)}{dx} = \pi \frac{dx}{dx} = \pi \cdot 1 = \pi.$$

Example: Find $\frac{dy}{dx}$ if $y = 1$.

$$\frac{dy}{dx} = \frac{d1}{dx} = \frac{d(1 \cdot 1)}{dx} = 1 \cdot \frac{d1}{dx} + \frac{d1}{dx} \cdot 1 = \frac{d1}{dx} + \frac{d1}{dx}.$$

Subtract $\frac{d1}{dx}$ from both sides.

So $\quad \frac{d1}{dx} = 0.$

Example: Find $\frac{dy}{dx}$ if $y = 5$.

$$\frac{dy}{dx} = \frac{d5}{dx} = \frac{d(5 \cdot 1)}{dx} = 5 \cdot \frac{d1}{dx} = 5 \cdot 0 = 0.$$

Example: Find $\frac{dy}{dx}$ if $y = 6342$.

$$\frac{dy}{dx} = \frac{d\,6342}{dx} = \frac{d(6342 \cdot 1)}{dx} = 6342 \cdot \frac{d1}{dx} = 6342 \cdot 0 = 0.$$

Example: Find $\frac{dc}{dx}$ if c is a constant.

$$\frac{dc}{dx} = \frac{d(c \cdot 1)}{dx} = c \cdot \frac{d1}{dx} = c \cdot 0 = 0.$$

Example: Find $\frac{dy}{dx}$ if $y = 3x + 12$.

$$\frac{dy}{dx} = \frac{d(3x + 12)}{dx} = \frac{d(3x)}{dx} + \frac{d(12)}{dx} = 3\frac{dx}{dx} + 0 = 3 \cdot 1 + 0 = 3.$$

Example: Find $\frac{dy}{dx}$ if $y = x^2$.

$$\frac{dy}{dx} = \frac{dx^2}{dx} = \frac{d(x \cdot x)}{dx} = x \frac{dx}{dx} + \frac{dx}{dx} x = x \cdot 1 + 1 \cdot x = 2x.$$

Example: Find $\frac{dy}{dx}$ if $y = x^3$.

$$\frac{dy}{dx} = \frac{dx^3}{dx} = \frac{d(x^2 \cdot x)}{dx} = x^2 \frac{dx}{dx} + \frac{dx^2}{dx} x = x^2 \cdot 1 + 2x \cdot x = 3x^2.$$

Example: Find $\dfrac{dy}{dx}$ if $y = x^4$.

$$\frac{dy}{dx} = \frac{dx^4}{dx} = \frac{d(x^3 \cdot x)}{dx} = x^3 \frac{dx}{dx} + \frac{dx^3}{dx} x = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3.$$

... and we keep on going ...

Example: Find $\dfrac{dy}{dx}$ if $y = x^{6342}$.

$$\frac{dy}{dx} = \frac{dx^{6342}}{dx} = \frac{d(x^{6341} \cdot x)}{dx} = x^{6341} \frac{dx}{dx} + \frac{dx^{6341}}{dx} x = x^{6341} \cdot 1 + 6341x^{6340} \cdot x = 6342x^{6341}.$$

... and we keep on going ...

Example: Find $\dfrac{dx^n}{dx}$ for $n = 1, 2, 3, \ldots$.

$$\begin{aligned}\frac{dy}{dx} = \frac{dx^n}{dx} = \frac{d(x^{n-1} \cdot x)}{dx} &= x^{n-1} \frac{dx}{dx} + \frac{dx^{n-1}}{dx} x \\ &= x^{n-1} \cdot 1 + (n-1)x^{n-2} \cdot x, \qquad \text{since we already found } \frac{dx^{n-1}}{dx} = (n-1)x^{n-2}, \\ &= nx^{n-1}.\end{aligned}$$

and thus we have found $\quad \dfrac{dx^n}{dx} = nx^{n-1}, \quad$ for all positive integers n. (Amazing!)

Example: Find $\dfrac{dx^n}{dx}$ for $n = 0$.

$$\frac{dy}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0 = 0x^{-1} = 0x^{0-1}.$$

Example: Find $\dfrac{dx^{-6342}}{dx}$.

$$\frac{dx^{-6342} \cdot x^{6342}}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dx^{-6342} \cdot x^{6342}}{dx} = x^{-6342} \frac{dx^{6342}}{dx} + \frac{dx^{-6342}}{dx} \cdot x^{6342} = x^{-6342} \cdot 6342x^{6341} + \frac{dx^{-6342}}{dx} \cdot x^{6342}.$$

So

$$0 = x^{-6342} \cdot 6342x^{6341} + \frac{dx^{-6342}}{dx} \cdot x^{6342}.$$

Solve for $\frac{dx^{-6342}}{dx}$.

$$\frac{dx^{-6342}}{dx} = -6342x^{-1}x^{-6342} = (-6342)x^{-6343}.$$

Example: Find $\frac{dx^{-n}}{dx}$ for $n = 1, 2, 3, \ldots$.

$$\frac{dx^{-n} \cdot x^n}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dx^{-n} \cdot x^n}{dx} = x^{-n}\frac{dx^n}{dx} + \frac{dx^{-n}}{dx} \cdot x^n = x^{-n} \cdot nx^{n-1} + \frac{dx^{-n}}{dx} \cdot x^n.$$

So

$$0 = x^{-n} \cdot nx^{n-1} + \frac{dx^{-n}}{dx} \cdot x^n.$$

Solve for $\frac{dx^{-n}}{dx}$.

$$\frac{dx^{-n}}{dx} = -nx^{-1}x^{-n} = (-n)x^{-n-1}.$$

and thus we have found $\quad \frac{dx^n}{dx} = nx^{n-1}, \quad$ for *all* integers n. (AMAZING!)

Example: Let $y = 3x^3 + 5x^2 + 2x + 7$. Find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(3x^3 + 5x^2 + 2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2 + 2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2)}{dx} + \frac{d(2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2)}{dx} + \frac{d(2x)}{dx} + \frac{d7}{dx} \\
&= 3\,\frac{dx^3}{dx} + 5\,\frac{dx^2}{dx} + 2\,\frac{dx}{dx} + 7\,\frac{d1}{dx} \\
&= 3 \cdot 3x^2 + 5 \cdot 2x + 2 \cdot 1 + 7 \cdot 0 = 9x^2 + 10x + 2.
\end{aligned}$$

Example: Let $y = -7x^{-13} + 5x^{-7} + (6 + 2i)x^{38}$. Find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(-7x^{-13} + 5x^{-7} + (6+2i)x^{38})}{dx} \\
&= \frac{d(-7x^{-13})}{dx} + \frac{d(5x^{-7})}{dx} + \frac{d((6+2i)x^{38})}{dx} \\
&= -7\,\frac{dx^{-13}}{dx} + 5\,\frac{dx^{-7}}{dx} + (6+2i)\,\frac{dx^{38}}{dx} \\
&= (-7)\cdot(-13)x^{-13-1} + 5(-7)x^{-7-1} + (6+2i)\cdot 38 \cdot x^{38-1} \\
&= 91x^{-14} - 35x^{-8} + (228+76i)x^{37}.
\end{aligned}$$