The chain rule

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There are **different kinds of derivatives**:

Derivative with respect to x

$$f \longrightarrow \tfrac{d}{dx} \longrightarrow \tfrac{df}{dx}$$

This one satisfies

$$\frac{dx}{dx} = 1,$$
$$\frac{d(cf)}{dx} = c\frac{df}{dx}, \quad \text{if } c \text{ is a constant,}$$
$$\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx},$$
$$\frac{d(yz)}{dx} = y\,\frac{dz}{dx} + \frac{dy}{dx}\,z.$$

Derivative with respect to g

$$f \longrightarrow \tfrac{d}{dg} \longrightarrow \tfrac{df}{dg}$$

This one satisfies

$$\frac{dg}{dg} = 1,$$
$$\frac{d(cf)}{dg} = c\frac{df}{dg}, \quad \text{if } c \text{ is a constant,}$$
$$\frac{d(y+z)}{dg} = \frac{dy}{dg} + \frac{dz}{dg},$$
$$\frac{d(yz)}{dg} = y\,\frac{dz}{dg} + \frac{dy}{dg}\,z.$$

What is the relation between $\dfrac{df}{dx}$ **and** $\dfrac{df}{dg}$ **?**

$$\longrightarrow \frac{d}{dg} \longrightarrow \qquad\qquad \longrightarrow \frac{d}{dx} \longrightarrow$$

$$\frac{dg^0}{dg} = \frac{d1}{dg} = 0, \qquad\qquad \frac{dg^0}{dx} = \frac{d1}{dx} = 0,$$

$$\frac{dg}{dg} = 1, \qquad\qquad \frac{dg}{dx} = \frac{dg}{dx},$$

$$\begin{aligned} \frac{dg^2}{dg} &= \frac{dg \cdot g}{dg} \\ &= g\frac{dg}{dg} + \frac{dg}{dg}g \\ &= g + g = 2g, \end{aligned} \qquad\qquad \begin{aligned} \frac{dg^2}{dx} &= \frac{dg \cdot g}{dx} \\ &= g\frac{dg}{dx} + \frac{dg}{dx}g \\ &= 2g\frac{dg}{dx}, \end{aligned}$$

$$\begin{aligned} \frac{dg^3}{dg} &= \frac{dg^2 \cdot g}{dg} \\ &= g^2\frac{dg}{dg} + \frac{dg^2}{dg}g \\ &= g^2 + 2g \cdot g = 3g^2, \end{aligned} \qquad\qquad \begin{aligned} \frac{dg^3}{dx} &= \frac{dg^2 \cdot g}{dx} \\ &= g^2\frac{dg}{dx} + \frac{dg^2}{dx}g \\ &= g^2\frac{dg}{dx} + 2g\frac{dg}{dx}g \\ &= g^2\frac{dg}{dx} + 2g^2\frac{dg}{dx} \\ &= 3g^2\frac{dg}{dx}, \end{aligned}$$

$$\begin{aligned} \frac{dg^4}{dg} &= \frac{dg^3 \cdot g}{dg} \\ &= g^3\frac{dg}{dg} + \frac{dg^3}{dg}g \\ &= g^3 + 3g^2 \cdot g = 4g^3, \end{aligned} \qquad\qquad \begin{aligned} \frac{dg^4}{dx} &= \frac{dg^3 \cdot g}{dx} \\ &= g^3\frac{dg}{dx} + \frac{dg^3}{dx}g \\ &= g^3\frac{dg}{dx} + 3g^2\frac{dg}{dx}g \\ &= g^3\frac{dg}{dx} + 3g^3\frac{dg}{dx} \\ &= 4g^3\frac{dg}{dx}, \end{aligned}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{dg^{6342}}{dg} = 6342g^{6341}, \qquad\qquad \frac{dg^{6342}}{dx} = 6342g^{6341}\frac{dg}{dx},$$

$$\begin{aligned}\frac{d(3g^2+2g+7)}{dg} &= \frac{d(3g^2)}{dg}+\frac{d(2g)}{dg}+\frac{d7}{dg}\\ &= 3\frac{dg^2}{dg}+2\frac{dg}{dg}+0\\ &= 3\cdot 2g+2\cdot 1\\ &= 6g+2,\end{aligned}$$

$$\begin{aligned}\frac{d(3g^2+2g+7)}{dx} &= \frac{d(3g^2)}{dx}+\frac{d(2g)}{dx}+\frac{d7}{dx}\\ &= 3\frac{dg^2}{dx}+2\frac{dg}{dx}+0\\ &= 3\cdot 2g\frac{dg}{dx}+2\frac{dg}{dx}\\ &= (6g+2)\frac{dg}{dx},\end{aligned}$$

Thus, we are seeing that

$$\frac{df}{dx}=\frac{df}{dg}\,\frac{dg}{dx}\,, \qquad \text{which is the chain rule.}$$

Example: Find $\frac{dy}{dx}$ when $y=(2x-5)^2$.

If $g=2x-5$ then $y=g^2$.

$$\frac{dy}{dx}=\frac{dy}{dg}\,\frac{dg}{dx}=\frac{dg^2}{dg}\,\frac{d(2x-5)g}{dx}=2g(2-0)=2(2x-5)\cdot 2$$

$$=4(2x-5)=8x-20.$$

Example: Find $\frac{dy}{dx}$ when $y=(3x-4)^3$.

If $g=3x-4$ then $y=g^3$.

$$\frac{dy}{dx}=\frac{dy}{dg}\,\frac{dg}{dx}=\frac{dg^3}{dg}\,\frac{d(3x-4)g}{dx}=3g^2(3-0)=9(3x-4)^2$$

$$=9(9x^2-24x+16)=81x^2-72x+144.$$

Example: Find $\frac{dy}{dx}$ when $y=(2x-5)^2(3x-4)^3$.

$$\frac{dy}{dx}=\frac{d(2x-5)^2(3x-4)^3}{dx}=(2x-5)^2\,\frac{d(3x-4)^3}{dx}+\frac{d(2x-5)^2}{dx}\,(3x-4)^3$$

$$=(2x-5)^2\cdot 3(3x-4)^2\cdot 3+2(2x-5)\cdot 2(3x-4)^3$$

$$=(2x-5)(3x-4)^2(9(2x-5)+4(3x-4))=(2x-5)(3x-4)^2(30x-61).$$

Example: Find $\frac{d\,x^{m/n}}{dx}$ when m and n are integers, $n\neq 0$.

$$\frac{d(x^{m/n})^n}{dx} = \frac{dx^m}{dx} = mx^{m-1}. \qquad \text{On the other hand} \qquad \frac{d(x^{m/n})^n}{dx} = n\big(x^{m/n}\big)^{n-1}\frac{dx^{m/n}}{dx}.$$

So $\quad mx^{m-1} = n\big(x^{m/n}\big)^{n-1}\dfrac{dx^{m/n}}{dx}$ $\quad$ and we can solve for $\dfrac{dx^{m/n}}{dx}$.

$$\frac{dx^{m/n}}{dx} = \frac{mx^{m-1}}{n\big(x^{m/n}\big)^{n-1}} = \frac{mx^{m-1}}{n\big(x^{m/n}\big)^{n}\big(x^{m/n}\big)^{-1}}$$

$$= \frac{mx^{m-1}}{nx^m\ \frac{1}{x^{m/n}}} = \left(\frac{m}{n}\right)x^{-1}x^{m/n} = \left(\frac{m}{n}\right)x^{(m/n)-1}.$$

Example: Find $\dfrac{dy}{dx}$ when $y = \dfrac{x}{\sqrt{1-2x}}$.

$$\frac{dy}{dx} = \frac{d\ \dfrac{x}{\sqrt{1-2x}}}{dx} = \frac{d\ x\big(\sqrt{1-2x}\big)^{-1}}{dx} = \frac{d\ x\big((1-2x)^{1/2}\big)^{-1}}{dx}$$

$$= \frac{d\ x(1-2x)^{-(1/2)}}{dx} = x\,\frac{d\ (1-2x)^{-(1/2)}}{dx} + \frac{dx}{dx}\,(1-2x)^{-(1/2)}$$

$$= x\big(-\tfrac{1}{2}\big)(1-2x)^{-3/2}\,\frac{d\ (1-2x)}{dx} + 1\cdot\frac{1}{\sqrt{1-2x}}$$

$$= \frac{-x}{2(1-2x)^{3/2}}\cdot(-2) + \frac{1}{(1-2x)^{1/2}} = \frac{x+1-2x}{(1-2x)^{3/2}} = \frac{1-x}{(1-2x)^{3/2}}.$$

Example: Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$.

$$\frac{dy}{dx} = \frac{d\,\dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}}{dx} = \frac{d\,\dfrac{(1+x^2)^{1/2}}{(1-x^2)^{1/2}}}{dx} = \frac{d\left(\dfrac{1+x^2}{1-x^2}\right)^{1/2}}{dx}$$

$$= \frac{1}{2}\cdot\left(\frac{1+x^2}{1-x^2}\right)^{(1/2)-1}\frac{d\left(\dfrac{1+x^2}{1-x^2}\right)}{dx}$$

$$= \frac{1}{2}\cdot\left(\frac{1+x^2}{1-x^2}\right)^{-(1/2)}\frac{d\,(1+x^2)(1-x^2)^{-1}}{dx}$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left((1+x^2)\frac{d\,(1-x^2)^{-1}}{dx} + \frac{d\,(1+x^2)}{dx}(1-x^2)^{-1}\right)$$

$$= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left((1+x^2)(-1)(1-x^2)^{-2}\frac{d\,(1-x^2)^{-1}}{dx} + 2x(1-x^2)^{-1}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left(\frac{(-1)(1+x^2)(-2x)}{(1-x^2)^2} + \frac{2x}{1-x^2}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left(\frac{2x(1+x^2)}{(1-x^2)^2} + \frac{2x(1-x^2)}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left(\frac{2x(1+x^2+1-x^2)}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{(1+x^2)^{1/2}}\cdot\frac{4x}{(1-x^2)^2} = \frac{2x}{(1+x^2)^{1/2}(1-x^2)^{3/2}}.$$

Example: Differentiate $\dfrac{x^2}{1+x^2}$ with respect to x^2.

This is the same problem as:

Find $\dfrac{dz}{dp}$ when $z = \dfrac{x^2}{1+x^2}$ and $p = x^2$.

Since $\dfrac{dz}{dx} = \dfrac{dz}{dp}\dfrac{dp}{dx}$, $\qquad \dfrac{dz}{dp} = \dfrac{(dz/dx)}{(dp/dx)}$.

So

$$\frac{dz}{dp} = \frac{\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)}{\frac{d}{dx}(x^2)} = \frac{\frac{d\ x^2(1+x^2)^{-1}}{dx}}{\frac{d\ x^2}{dx}} = \frac{x^2\frac{d\ (1+x^2)^{-1}}{dx} + \frac{dx^2}{dx}(1+x^2)^{-1}}{2x}$$

$$= \frac{x^2(-1)(1+x^2)^{-2}\frac{d\ (1+x^2)}{dx} + 2x(1+x^2)^{-1}}{2x}$$

$$= \frac{\frac{-x^2}{(1+x^2)^2}\cdot 2x + \frac{2x}{1+x^2}}{2x} = \frac{-x^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{-x^2+1+x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2}.$$

Example: Find $\frac{dy}{dx}$ when $x^4+y^4 = 4a^2x^2y^2$.

$$\frac{d\,(x^4+y^4)}{dx} = \frac{d\,(4a^2x^2y^2)}{dx}. \qquad \text{So} \quad \frac{dx^4}{dx} + \frac{dy^4}{dx} = 4a^2\frac{dx^2y^2}{dx}.$$

So $\quad 4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2\frac{dy^2}{dx} + \frac{dx^2}{dx}y^2\right).$

So $\quad 4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2 2y\frac{dy}{dx} + 2xy^2\right)$

$$= 4a^2x^2 2y\frac{dy}{dx} + 4a^2 2xy^2.$$

So $\quad 4x^3 - 4a^2 2xy^2 = 4a^2x^2 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}.$

So $\quad 4x^3 - 4a^2 2xy^2 = \left(4a^2x^2 2y - 4y^3\right)\frac{dy}{dx}.$

So $\quad \frac{4x^3 - 4a^2 2xy^2}{4a^2x^2 2y - 4y^3} = \frac{dy}{dx}.$

So $\quad \frac{dy}{dx} = \frac{x^3 - 2a^2xy^2}{2a^2x^2y - y^3}.$

All we did is take the derivative of both sides and then solve for $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ when $x = \frac{3at}{1+t^3}$ and $y = \frac{3at^2}{1+t^3}$.

Since $y = \frac{3at^2}{1+t^3} = \left(\frac{3at}{1+t^3}\right)t = xt, \qquad \frac{dy}{dx} = x\frac{dt}{dx} + \frac{dx}{dx}\cdot t = x\frac{dt}{dx} + t.$

What is $\dfrac{dt}{dx}$??

Since $\dfrac{dx}{dx} = \dfrac{dx}{dt}\,\dfrac{dt}{dx}$, $\qquad \dfrac{dt}{dx} = \dfrac{\big(dx/dx\big)}{\big(dx/dt\big)} = \dfrac{1}{dx/dt}$.

So

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{\dfrac{d}{dt}\left(\dfrac{3at}{1+t^3}\right)} = \frac{1}{\dfrac{d\ (3at)(1+t^3)^{-1}}{dt}}$$

$$= \frac{1}{3at(-1)(1+t^3)^{-2}\dfrac{d(1+t^3)}{dt} + 3a(1+t^3)^{-1}}$$

$$= \frac{1}{\dfrac{-3at}{(1+t^3)^2}\,3t^2 + \dfrac{3a}{1+t^3}} = \frac{1}{\dfrac{-9at^3 + 3a(1+t^3)}{(1+t^3)^2}}$$

$$= \frac{(1+t^3)^2}{-9at^3 + 3a(1+t^3)} = \frac{(1+t^3)^2}{3a - 6at^3}.$$

So

$$\frac{dy}{dx} = x\frac{dt}{dx} + t = \frac{3at}{1+t^3}\,\frac{(1+t^3)^2}{3a(1-2t^3)} + t$$

$$= \frac{t(1+t^3)}{1-2t^3} + \frac{t(1-2t^3)}{1-2t^3} = \frac{t + t^4 + t - 2t^4}{1-2t^3} = \frac{2t - t^4}{1-2t^3}$$