Quantization

1. h-adic topology

1.1 Let A be a ring. Let $\mathfrak{a}$ be an ideal in A. We view the powers $\mathfrak{a}^k$ of the ideal $\mathfrak{a}$ as a basis of neighborhoods in A containing 0. There is a unique topology on A such that the ring operations are continuous with a basis given by the sets $a + \mathfrak{a}^k$ where $a \in A$. This is the $\mathfrak{a}$-*adic topology.* If $\bigcap_k \mathfrak{a}^k = (0)$ then this topology is Hausdorff.

1.2 Let M be an A-module. We can transfer the $\mathfrak{a}$-adic topology on A to a topology on M. We view the sets $N_k = \mathfrak{a}^k M$ as a basis of neighborhoods in M containing 0. An element $m \in M$ is an element of N_k if $m = 0 \pmod{\mathfrak{a}^k M}$. As above, there is a unique topology on M such that the module operations are continuous with basis given by the sets $m + \mathfrak{a}^k M$ where $m \in M$. This is the $\mathfrak{a}$-*adic topology* on M.

1.3 Define a map $d: M \times M \to \mathbb{R}$ by

$$d(x, y) = e^{-v(x-y)}, \qquad \text{for all } x, y \in M,$$

where e is a real number $e > 1$ and $v(x)$ is the largest integer k such that $x \in \mathfrak{a}^k M$. If the $\mathfrak{a}$-adic topology on M is Hausdorff then d is a metric on M which generates the $\mathfrak{a}$-adic topology.

1.4 If A is a local ring then it is natural to take $I = \mathfrak{m}$ where $\mathfrak{m}$ is the unique maximal ideal in A. If k is a field and h is an indeterminate then the ring of formal power series in h, $k[[h]]$, is a local ring with unique maximal ideal $\mathfrak{m} = (h)$ generated by h. In this case the $\mathfrak{m}$-adic topology on a $k[[h]]$ module M is called the h-*adic topology* on M.

1.5 Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Let M be an A-module. A sequence of elements $\{b_n\}$ in M is a *Cauchy sequence* in the $\mathfrak{a}$-adic topology if for every positive integer $k > 0$ there exists a positive integer N such that

$$b_n - b_m \in \mathfrak{a}^k M \text{ for all } m, n > N.$$

A sequence $\{b_n\}$ of elements in M *converges to* $b \in M$ if for every positive integer $k > 0$ there exists a positive integer N such that

$$b_n - b \in \mathfrak{a}^k M \text{ for all } n > N.$$

The module M is *complete* in the $\mathfrak{a}$-adic topology if every Cauchy sequence in M converges. A ring A is complete in the $\mathfrak{a}$-adic topology if when viewed as an A-module it is complete in the $\mathfrak{a}$-adic topology. If the $\mathfrak{a}$-adic topology is Hausdorff then this definition of completeness is the same as the ordinary definition of completeness when we view that M as a metric space as in (1.1).

1.6 Two Cauchy sequences $P = \{p_n\}$ and $Q = \{q_n\}$ in M are *equivalent* if $\{p_n - q_n\}$ converges to 0 in the $\mathfrak{a}$-adic topology, i.e.,

$$P \sim Q \text{ if for every } k \text{ there exists an } N \text{ such that } p_n - q_n \in \mathfrak{a}^k M \text{ for all } n > N.$$

The set of all equivalence classes of Cauchy sequences in M is the *completion* $\hat{M}$ of M.

1.7 The completion $\hat{M}$ is an $\hat{A}$ module with operations given by

$$\begin{aligned} P + Q &= \{p_n + q_n\}, \qquad \text{and,} \\ aP &= \{a_n p_n\}, \end{aligned}$$

where $P = \{p_n\}$ and $Q = \{q_n\}$ are Cauchy sequences with elements in M and $\{a_n\}$ is a Cauchy sequence of elements in A.

1.8 Define a map $\phi: M \to \hat{M}$ given by

$$\phi(b) = [(b, b, b, \ldots)],$$

i.e. $\phi(b)$ is the equivalence class of the sequence $\{b_n\}$ such that $b_n = b$ for all n. This map has kernel $\bigcap_k \mathfrak{a}^k M$. The map ϕ is injective if M is Hausdorff in the $\mathfrak{a}$-adic topology.

1.9 Define a basis N_k of neighborhoods of 0 in the completion $\hat{M}$ by:

$P \in N_k$ if there exists an N such that $p_n \in \mathfrak{a}^k M$ for all $n > N$.

The collection of sets $P + N_k$ where $P \in \hat{M}$ is a basis for a topology on $\hat{M}$. The module operations and the map ϕ are continuous.

1.10 Let k be a field. Then $k[[h]]$ is a local ring with maximal ideal $\mathfrak{m} = (h)$ generated by the element h. In this case the $\mathfrak{m}$-adic topology is called the h-adic topology. Let M be a $k[[h]]$-module. Then a sequence of elements $\{b_n\}$ in M is a *Cauchy sequence* if for every positive integer $k > 0$ there exists a positive integer N such that

$$b_n - b_m \in h^k M \text{ for all } m, n > N,$$

i.e., $b_n - b_m$ is "divisible" by h^k for all $n, m > N$. A sequence $\{b_n\}$ of elements in $k[[h]]$ *converges to* $b \in M$ if for every positive integer $k > 0$ there exists a positive integer N such that

$$b_n - b \in h^k M \text{ for all } n > N.$$

The module M is *complete* in the h-adic topology if every Cauchy sequence in M converges.

1.11 As in (1.2) we can define the *completion* of a $k[[h]]$-module M in the h-adic topology. If A is an algebra over a field k then $A \otimes_k k[[h]]$ is a $k[[h]]$-module and the completion of $A \otimes_k k[[h]]$ is $A[[h]]$, the ring of formal power series in h with coefficients in A. The ring $A[[h]]$ is, in general, larger than $A \otimes_k k[[h]]$.

1.12 If M is complete $k[[h]]$-module in the h-adic topology then for each element $x = \sum_{j \geq 0} x_j h^j \in M$ the element

$$e^{hx} = \sum_{k \geq 0} \frac{(hx)^k}{k!} = 1 + x_0 h + (x_0^2 + 2x_1)\left(\frac{h^2}{2}\right) + (x_0^3 + 3(x_0 x_1 + x_1 x_0) + 6x_2)\left(\frac{h^3}{3!}\right) + \cdots$$

is a well defined element of M.

1.13 A $k[[h]]$-module M is *topologically free* if $M/h^k M$ is a free $k[[h]]/(h^k)$-module for all positive integers $k > 0$.

2. Deformations and quantizations

2.1 A *deformation* of a commutative associative algebra over k is an associative (not necessarily commutative) algebra A over $k[[h]]$ such that

1) $A/hA = A_0$, and
2) A is a topologically free $k[[h]]$ module.

2.2 Given a deformation A of a commutative algebra A_0 we can define a new operation $\{,\}$ on A_0 by defining

$$\{a \bmod h, b \bmod h\} = \frac{[a,b]}{h} \bmod h,$$

where $[a,b] = ab - ba$. This makes A_0 into a Poisson algebra. If A_0 was a Poisson algebra to start with then we would like this new Poisson structure to be the same as the old one.

2.3 Dualize the above definitions to define a deformation of a cocommutative Poisson algebra. Then extend the picture to co-Poisson-Hopf algebras. This is the motivation for the following definition.

Let $(\mathfrak{g}, \phi)$ be a Lie bialgebra and let $\delta\colon \mathfrak{Ug} \to \mathfrak{Ug} \otimes \mathfrak{Ug}$ the corresponding Poisson cobracket. A *quantization* of $(\mathfrak{g}, \phi)$ is a topological Hopf algebra (A, Δ) over $k[[h]]$ which is a topologically free $k[[h]]$-module and satisfies the following conditions:

1) A/hA is identical with $\mathfrak{Ug}$ as a Hopf algebra, and
2) (Co-Poisson compatibility)

$$h^{-1}(\Delta(a) - \sigma(\Delta(a))) \bmod h = \delta(a \bmod h)$$

for $a \in A$, where $\sigma\colon A \otimes A$ and $\sigma(x \otimes y) = y \otimes x$.

2.4 The definition of deformations given in (2.1) is too general for some purposes. Assume that A is an algebra over a field k with multiplication $m\colon A \otimes A \to A$. Let h be a indeterminate and let $A[[h]]$ be the ring of formal power series in h with coefficients in A. This is a complete $k[[h]]$ module. A *deformation* of A is an associative $k[[h]]$ bilinear multiplication $m_h\colon A[[h]] \otimes_{k[[h]]} A[[h]] \to A[[h]]$ map which can be written in the form

$$m_h = m + m_1 h + m_2 h^2 + \cdots,$$

where the $m_i\colon A \otimes A \to A$ are k-linear maps which are extended to the completion $A[[h]]$.

If A is a bialgebra over k with multiplication $m\colon A \otimes A \to A$ and comultiplication $\Delta\colon A \to A \otimes A$ then a *deformation* of A is a $k[[h]]$-linear multiplication and a $k[[h]]$-linear comultiplication

$$\begin{aligned} m_h &= m + m_1 h + m_2 h^2 + \cdots, \\ \Delta_h &= \Delta + \Delta_1 h + \Delta_2 h^2 + \cdots, \end{aligned}$$

such that $m_i\colon A \otimes A \to A$ and $\Delta_i\colon A \to A \otimes A$ are k-linear maps which are extended to the completion $A[[h]]$, and such that $A[[h]]$ is a bialgebra under m_h and Δ_h.

2.5 Suppose that

$$\begin{aligned} m_h &= m + m_1 h + m_2 h^2 + \cdots, \quad \text{and} \\ \mu_h &= m + \mu_1 h + \mu_2 h^2 + \cdots, \end{aligned}$$

are both deformations of an algebra A over k with multiplication m. The two deformations m_h and μ_h are *equivalent* if there is a $k[[h]]$-linear map $f_t\colon A[[h]] \to A[[h]]$ of the form $f_t = id + f_1 h + f_2 h^2 + \cdots$, such that the $f_i\colon A \otimes A \to A$ are k-linear maps extended to the completion $A[[h]]$ such that

$$f_h \circ m_h(a \otimes b) = \mu_h(f_h(a) \otimes f_h(b))$$

for all $a, b \in A[[h]]$.

3. References

Drinfel'd has completely formalized the quantization process in the following paper in which he also introduced the object which is now called the Drinfel'd-Jimbo quantum group.

[D1] V.G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985) 254-258.

Concerning open problems in the theory of quantization:

[D3] V.G. Drinfel'd, *On some unsolved problems in quantum group theory*, in "Quantum Groups" Proceedings of the Euler International Mathematical Institute, Leningrad, Springer Lect. Notes. 1510, P. Kulish Ed., (1991) 1-8.

The following books have discussions of h-adic topology and completions. The definitions of completion for a metric space are found in Rudin's elementary analysis book Chapt. 3 Exercise 23-24.

[AM] M.F. Atiyah and I.G. Macdonald, "Introduction to Commutative algebra", Addison-Wesley, 1969.

[ZS] O. Zariski and P. Samuel, "Commutative algebra", Vol. II, Van-Nostrand 1960.

Deformation theory was developed by Gerstenhaber in a series of papers in Annals of Mathematics 1964-74. More recently this theory has been developing in the context of quantum groups, in particular see [GGS] and the references there.

[GGS] M. Gerstenhaber, A. Giaquinto, and S. Schack, *Quantum symmetry*, Proceedings of Workshops in the Euler Int. Math. Inst., Leningrad 1990, Springer Lecture Notes No. 1510, P.P. Kulish Ed., (1991) 9-46.

[Sn] S. Shnider, *Deformation cohomology for bialgebras and quasi-bialgebras*, Contemporary Mathematics **134** Amer. Math. Soc. (1992) 259-296.