The q-Weyl dimension formula

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville VIC 3010 Australia
aram@unimelb.edu.au

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1 The Weyl dimension formula and its q-analogue

This material is taken from notes for a joint paper with Zajj Daugherty and Rahbar Virk. Everything here is "well known".

Proposition 1.1. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra etc.etc. ...???? Let ν be a dominant integral weight so that the irreducible module $L(\nu)$ of highest weight ν is finite dimensional. Let $\hat{\kappa}_1 = \langle \nu, \nu + 2\rho \rangle$.*

(a) $(\mathrm{id} \otimes \mathrm{tr}_{L(\nu)})((\frac{1}{2}\hat{\kappa}_1 + \gamma)^0) = \dim(L(\nu)) = \mathrm{ev}_0(s_\nu) = \prod_{\alpha \in R^+} \frac{\langle \nu + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$,

(b) $(\mathrm{id} \otimes \mathrm{tr}_{L(\nu)})((\frac{1}{2}\hat{\kappa}_1 + \gamma)^1) = \frac{1}{2}\hat{\kappa}_1 \dim(L(\nu))$.

(c) $(\mathrm{id} \otimes \mathrm{qtr}_{L(\nu)})((\mathcal{R}_{21}\mathcal{R})^0) = \dim_q(L(\nu)) = \mathrm{ev}_{2\rho}(s_\nu) = \prod_{\alpha \in R^+} \frac{[\langle \nu + \rho, \alpha \rangle]}{[\langle \rho, \alpha \rangle]}$,

(d) [TW, Lemma 3.5.1] $(\mathrm{id} \otimes \mathrm{qtr}_{L(\nu)})((\mathcal{R}_{21}\mathcal{R})^1)$ *acts on* $L(\mu)$ *by* $\mathrm{ev}_{2(\mu+\rho)}(s_\nu)\mathrm{id}_{L(\mu)}$.

Proof. (a) and (c) Let M be a U-module. On $M \otimes L(\nu)$,

$$(\mathrm{id} \otimes \mathrm{qtr}_{L(\nu)})((\mathcal{R}_{21}\mathcal{R})^0) = (\mathrm{id} \otimes \mathrm{qtr}_{L(\nu)})(\mathrm{id}_{M \otimes L(\nu)}) = \mathrm{id}_M \dim_q(L(\nu)),$$

giving the first identity in (c). By (**??**) and the second identity in (**??**), $v = e^{-h\rho}u$, and

$$\begin{aligned}\dim_q(L(\nu))\mathrm{id}_M &= (\mathrm{id} \otimes \mathrm{qtr}_{L(\nu)})(\mathrm{id}_{M \otimes L(\nu)}) = (\mathrm{id} \otimes \mathrm{tr}_{L(\nu)})(1 \otimes uv^{-1}) \\ &= (\mathrm{id} \otimes \mathrm{tr}_{L(\nu)})(1 \otimes e^{h\rho}) = \mathrm{ev}_{2\rho}(s_\nu)\mathrm{id}_M,\end{aligned}$$

since $q = e^{h/2}$. This establishes the second identity in (c) and the last equality in (c) is

$$\mathrm{ev}_{2\rho}(s_\nu) = \mathrm{ev}_{2\rho}\left(\frac{a_{\nu+\rho}}{a_\rho}\right) = \frac{\sum_{w\in W}\det(w)q^{\langle w(\nu+\rho),2\rho\rangle}}{\sum_{w\in W}\det(w)q^{\langle w\rho,2\rho\rangle}} = \frac{\sum_{w\in W}\det(w)q^{\langle 2(\nu+\rho),w\rho\rangle}}{\sum_{w\in W}\det(w)q^{\langle 2\rho,w\rho\rangle}}$$
$$= \frac{\mathrm{ev}_{2(\nu+\rho)}(a_\rho)}{\mathrm{ev}_{2\rho}(a_\rho)} = \prod_{\alpha\in R^+}\frac{\mathrm{ev}_{2(\nu+\rho)}(X^{\alpha/2}-X^{-\alpha/2})}{\mathrm{ev}_{2\rho}(X^{\alpha/2}-X^{-\alpha/2})} = \prod_{\alpha\in R^+}\frac{q^{\langle\nu+\rho,\alpha\rangle}-q^{-\langle\nu+\rho,\alpha\rangle}}{q^{\langle\rho,\alpha\rangle}-q^{-\langle\rho,\alpha\rangle}}$$
$$= \prod_{\alpha\in R^+}\frac{[\langle\nu+\rho,\alpha\rangle]}{[\langle\rho,\alpha\rangle]}.$$

The last equality in (a) is

$$\mathrm{ev}_0(s_\nu) = \lim_{q\to 1}\mathrm{ev}_{2\rho}(s_\nu) = \prod_{\alpha\in R^+}\frac{\langle\nu+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.$$

WHERE WAS THE q-analogue of k, $[k] = (q^k - q^{-k})(q - q^{-1})$, defined?

(b) Since $\pi_0((\mathrm{id}\otimes\mathrm{tr}_{L(\nu)})(\gamma))$ is a degree 1 element of $S(\mathfrak{h})$, and there are no W_0-invariants of degree 1 in $S(\mathfrak{h})$, then $\pi_0((\mathrm{id}\otimes\mathrm{tr}_{L(\nu)})(\gamma)) = 0$. Thus $\pi_0((\mathrm{id}\otimes\mathrm{tr}_{L(\nu)})(\frac{1}{2}\hat\kappa_1+\gamma)) = \frac{1}{2}\hat\kappa_1\dim(L(\nu))$. SO SINCE PI 0 OF THIS ELEMENT IS THAT CONSTANT, THIS ELEMENT ITSELF IS THAT CONSTANT

(d) Let $h_1,\ldots,h_r$ be an orthonormal basis of $\mathfrak{h}$. By [?, §4] (see [LR, (2.13)]) there is an expression

$$\mathcal{R} = e^{\frac{1}{2}h\gamma_0} + \sum b_j^+\otimes b_j^-, \qquad \text{where}\quad \gamma_0 = \sum_{\ell=1}^r h_\ell\otimes h_\ell,$$

and $b_j^+\in U^+$ and $b_j^-\in U^-$ are homogeneous elements of degree greater than 0. Let $v_1,\ldots,v_n$ be a basis of weight vectors of $L(\nu)$ and let $v^1,\ldots,v^n$ the dual basis in $L(\nu)^*$. Let v_μ^+ be a highest weight vector in $L(\mu)$. Since $b_j^+v_\mu^+ = 0$ and $q = e^{h/2}$, $\mathcal{R}_{21}\mathcal{R}\otimes 1$ acts on $v_\mu^+\otimes v_i\otimes v^i\in L(\mu)\otimes L(\nu)\otimes L(\nu)^*$ by

$$\begin{aligned}
&(\mathcal{R}_{21}\mathcal{R}\otimes\mathrm{id})(v_\mu^+\otimes v_i\otimes v^i)\\
&= \left(\Big(q^{\gamma_0}+\sum_j(b_j^-\otimes b_j^+)\Big)\Big(q^{\gamma_0}+\sum_j(b_j^+\otimes b_j^-)\Big)(v_\mu^+\otimes v_i)\right)\otimes v^i\\
&= \left(\Big(q^{\gamma_0}q^{\gamma_0}+q^{\gamma_0}\sum_j(b_j^-\otimes b_j^+)\Big)(v_\mu^+\otimes v_i)\right)\otimes v^i\\
&= \left(\Big(q^{2\sum_{\ell=1}^r\mu(h_\ell)\mathrm{wt}(v_i)(h_\ell)}+q^{\sum_{\ell=1}^r\mu(h_\ell)\mathrm{wt}(v_i)(h_\ell)}\sum_j(b_j^-\otimes b_j^+)\Big)\otimes(v_\mu^+\otimes v_i)\right)\otimes v^i\\
&= q^{2\langle\mu,\mathrm{wt}(v_i)\rangle}(v_\mu^+\otimes v_i\otimes v^i)+q^{\langle\mu,\mathrm{wt}(v_i)\rangle}\sum_j(b_j^-v_\mu^+\otimes b_j^+v_i\otimes v^i).
\end{aligned}$$

Since $(\mathrm{id}\otimes \mathrm{qtr}_{L(\nu)})(\mathcal{R}_{21}\mathcal{R})$ is central in U, it acts on v_μ^+ by a scalar. Therefore, since b_j^- is a lowering operator, and $uv^{-1} = u(e^{-h\rho^\vee}u)^{-1} = e^{h\rho^\vee}$,

$$\begin{aligned}
(\mathrm{id}\otimes \mathrm{qtr}_{L(\nu)})(\mathcal{R}_{21}\mathcal{R})v_\mu^+ &= \sum_i \langle v^i, uv^{-1}v_i\rangle q^{2\langle \mu, \mathrm{wt}(v_i)\rangle} v_\mu^+ + \sum_{i,j} q^{\langle \mu,\mathrm{wt}(v_i)\rangle}\langle v^i, uv^{-1}b_j^+ v_i\rangle b_j^- v_\mu^+ \\
&= \sum_i \langle v^i, e^{h\rho} v_i\rangle q^{2\langle \mu,\mathrm{wt}(v_i)\rangle} v_\mu^+ + 0 = \sum_i q^{\langle 2\rho, \mathrm{wt}(v_i)\rangle} q^{2\langle \mu,\mathrm{wt}(v_i)\rangle} v_\mu^+ \\
&= \sum_i q^{2\langle \mu+\rho, \mathrm{wt}(v_i)\rangle} v_\mu^+ = \mathrm{ev}_{2(\mu+\rho)}(s_\nu) v_\mu^+ \\
&= \frac{\sum_{w\in W_0} \det(w) q^{2\langle \mu+\rho, w(\nu+\rho)\rangle}}{\sum_{w\in W_0}\det(w) q^{2\langle \mu+\rho, w\rho\rangle}}.
\end{aligned}$$

□

2 Some q-dimension examples

Type B_r. The crystal for $L(\varepsilon_1)$ is

$$v_{\varepsilon_1} \xrightarrow{\tilde f_1} v_{\varepsilon_2} \xrightarrow{\tilde f_2} \cdots \xrightarrow{\tilde f_{r-1}} v_{\varepsilon_r} \xrightarrow{\tilde f_r} v_0 \xrightarrow{\tilde f_r} v_{-\varepsilon_r} \xrightarrow{\tilde f_{r-1}} \cdots \xrightarrow{\tilde f_2} v_{-\varepsilon_2} \xrightarrow{\tilde f_1} v_{-\varepsilon_1}$$

which indicates that

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \ldots, \quad \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r, \quad \alpha_r = \varepsilon_r,$$

so that

DYNKIN DIAGRAM WITH double bond connecting α_{r-1} and α_r with α_r short,

and

$$R^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1\le i<j\le r\} \cup \{\varepsilon_i \mid 1 \le i \le r\}.$$

Then

$$\rho = \frac{1}{2}\sum \alpha\in R^+ = \frac{1}{2}\left(\sum_{i=1}^r \varepsilon_i + \sum_{1\le i<j\le r} (\varepsilon_i - \varepsilon_j) + (\varepsilon_i+\varepsilon_j)\right) = \frac{1}{2}\sum_{i=1}^r (2r-2i+1)\varepsilon_i$$

satisfies $\langle \rho, \alpha_i^\vee\rangle = \langle \rho, \varepsilon_i - \varepsilon_{i+1}\rangle = 1$, for $i = 1, 2, \ldots, r-1$ and

$$\langle \rho, \alpha_r^\vee\rangle = \langle \rho, 2\varepsilon_r\rangle = 1.$$

Then the quantum dimension of $L(\varepsilon_1)$ is

$$\begin{aligned}
&q^{\langle 2\rho,\varepsilon_1\rangle} + \cdots + q^{\langle 2\rho, \varepsilon_r\rangle} + q^{\langle 2\rho, 0\rangle} + q^{\langle 2\rho, -\varepsilon_r\rangle} + \cdots + q^{\langle 2\rho, -\varepsilon_1\rangle} \\
&\quad = q^{2r-2+1} + q^{2r-4+1} + \cdots + q^{2r-2r+1} + q^0 + q^{-(2r-2r+1)} + \cdots + q^{-(2r-2+1)} \\
&\quad = q^{2r-1} + q^{2r-3} + \cdots + q + 1 + q^{-1} + \cdots + q^{-(2r-1)} = [2r]+1.
\end{aligned}$$

If $\lambda = \varepsilon_1$ then

$$\begin{aligned}
\prod_{\alpha\in R^+} \frac{[\langle\lambda+\rho,\alpha\rangle]}{[\langle\rho,\alpha\rangle]} &= \prod_{i=1}^{r} \frac{[\langle\lambda+\rho,\varepsilon_i\rangle]}{[\langle\rho,\varepsilon_i\rangle]} \prod_{1\le i<j\le r} \frac{[\langle\lambda+\rho,\varepsilon_i-\varepsilon_j\rangle]}{[\langle\rho,\varepsilon_i-\varepsilon_j\rangle]} \frac{[\langle\lambda+\rho,\varepsilon_i+\varepsilon_j\rangle]}{[\langle\rho,\varepsilon_i+\varepsilon_j\rangle]} \\
&= \prod_{i=1}^{r} \frac{[\lambda_i+r-i+\frac{1}{2})]}{[r-i+\frac{1}{2})]} \prod_{1\le i<j\le r} \frac{[\lambda_i-\lambda_j+j-i]}{[j-i]} \frac{[\lambda_i+\lambda_j+2r-i-j+1]}{[2r-i-j+1]} \\
&= \frac{[r+\frac{1}{2}]}{[r-\frac{1}{2}]} \prod_{j=2} \frac{[2][3]\cdots[r]}{[1][2]\cdots[r-1]} \frac{[2r-1][2r-2]\cdots[r+1]}{[2r-2][2r-3]\cdots[r]} \\
&= \frac{[r+\frac{1}{2}]}{[r-\frac{1}{2}]} \frac{[r]}{[1]} \frac{[2r-1]}{[r]} = \frac{[r+\frac{1}{2}][2r-1]}{[r-\frac{1}{2}][1]} = \frac{(q^{r+\frac{1}{2}}-q^{-(r+\frac{1}{2})})(q^{2r-1}-q^{-(2r-1)})}{(q^{r-\frac{1}{2}}-q^{-(r-\frac{1}{2})})(q-q^{-1})} \\
&= \frac{(q^{r+\frac{1}{2}}+q^{-(r+\frac{1}{2})})(q^{r-\frac{1}{2}}-q^{-(r-\frac{1}{2})})}{q-q^{-1}} = \frac{q^{2r}+q-q^{-1}-q^{-2r}}{q-q^{-1}} = [2r]+1.
\end{aligned}$$

Type C_r. The crystal for $L(\varepsilon_1)$ is

$$v_{\varepsilon_1} \xrightarrow{\tilde{f}_1} v_{\varepsilon_2} \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{r-1}} v_{\varepsilon_r} \xrightarrow{\tilde{f}_r} v_{-\varepsilon_r} \xrightarrow{\tilde{f}_{r-1}} \cdots \xrightarrow{\tilde{f}_2} v_{-\varepsilon_2} \xrightarrow{\tilde{f}_1} v_{-\varepsilon_1}$$

which indicates that

$$\alpha_1 = \varepsilon_1-\varepsilon_2, \quad \alpha_2 = \varepsilon_2-\varepsilon_3, \quad \ldots, \quad \alpha_{r-1} = \varepsilon_{r-1}-\varepsilon_r, \quad \alpha_r = 2\varepsilon_r,$$

so that

DYNKIN DIAGRAM WITH double bond connecting α_{r-1} and α_r with α_r long,

and

$$R^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1\le i<j\le r\} \cup \{2\varepsilon_i \mid 1\le i\le r\}.$$

Then

$$\rho = \frac{1}{2}\sum \alpha \in R^+ = \frac{1}{2}\left(\sum_{i=1}^{r} 2\varepsilon_i + \sum_{1\le i<j\le r} (\varepsilon_i-\varepsilon_j)+(\varepsilon_i+\varepsilon_j)\right) = \frac{1}{2}\sum_{i=1}^{r}(2r-2i+2)\varepsilon_i$$

satisfies $\langle\rho,\alpha_i^\vee\rangle = \langle\rho,\varepsilon_i-\varepsilon_{i+1}\rangle = 1$, for $i=1,2,\ldots,r-1$ and

$$\langle\rho,\alpha_r^\vee\rangle = \langle\rho,\varepsilon_r\rangle = 1.$$

Then the quantum dimension of $L(\varepsilon_1)$ is

$$\begin{aligned}
q^{\langle 2\rho,\varepsilon_1\rangle} &+ \cdots + q^{\langle 2\rho,\varepsilon_r\rangle} + q^{\langle 2\rho,-\varepsilon_r\rangle} + \cdots + q^{\langle 2\rho,-\varepsilon_1\rangle} \\
&= q^{2r-2+2} + q^{2r-4+2} + \cdots + q^{2r-2r+2} + q^{-(2r-2r+2)} + \cdots + q^{-(2r-2+2)} \\
&= q^{2r} + q^{2r-2} + \cdots + q^2 + q^{-2} + \cdots + q^{-2r} = [2r+1]-1.
\end{aligned}$$

If $\lambda = \varepsilon_1$ then

$$
\begin{aligned}
\prod_{\alpha \in R^+} \frac{[\langle \lambda+\rho, \alpha\rangle]}{[\langle \rho, \alpha\rangle]} &= \prod_{i=1}^{r} \frac{[\langle \lambda+\rho, 2\varepsilon_i\rangle]}{[\langle \rho, 2\varepsilon_i\rangle]} \prod_{1\le i<j\le r} \frac{[\langle \lambda+\rho, \varepsilon_i-\varepsilon_j\rangle]}{[\langle \rho, \varepsilon_i-\varepsilon_j\rangle]} \frac{[\langle \lambda+\rho, \varepsilon_i+\varepsilon_j\rangle]}{[\langle \rho, \varepsilon_i+\varepsilon_j\rangle]} \\
&= \prod_{i=1}^{r} \frac{[2(\lambda_i+r-i+1)]}{[2(r-i+1)]} \prod_{1\le i<j\le r} \frac{[\lambda_i-\lambda_j+j-i]}{[j-i]} \frac{[\lambda_i+\lambda_j+2r-i-j+2]}{[2r-i-j+2]} \\
&= \frac{[2+2r-2+2]}{[2r-2+2]} \prod_{j=2} \frac{[2][3]\cdots[r]}{[1][2]\cdots[r-1]} \frac{[2r][2r-1]\cdots[r+2]}{[2r-1][2r-2]\cdots[r+1]} \\
&= \frac{[2r+2]}{[2r]} \frac{[r]}{[1]} \frac{[2r]}{[r+1]} = \frac{[2r+2][r]}{[r+1][1]} = \frac{(q^{2r+2}-q^{-(2r+2)})(q^r-q^{-r})}{(q^{r+1}-q^{-(r+1)})(q-q^{-1})} \\
&= \frac{(q^{r+1}+q^{-(r+1)})(q^r-q^{-r})}{q-q^{-1}} = \frac{q^{2r+1}-q+q^{-1}-q^{-(2r+1)}}{q-q^{-1}} = [2r+1]-1.
\end{aligned}
$$

Type D_r. The crystal for $L(\varepsilon_1)$ is

$$
v_{\varepsilon_1} \xrightarrow{\tilde f_1} v_{\varepsilon_2} \xrightarrow{\tilde f_2} \cdots \xrightarrow{\tilde f_{r-2}} v_{\varepsilon_{r-1}}
\begin{array}{c} \nearrow \tilde f_{r-1} \quad v_{\varepsilon_r} \quad \tilde f_{r-1} \searrow \\ \tilde f_r \searrow \quad v_{-\varepsilon_r} \quad \nearrow \tilde f_r \end{array}
v_{-\varepsilon_{r-1}} \xrightarrow{\tilde f_{r-1}} v_{-\varepsilon_{r-2}} \xrightarrow{\tilde f_{r-2}} \cdots \xrightarrow{\tilde f_2} v_{-\varepsilon_2} \xrightarrow{\tilde f_1} v_{-\varepsilon_1}
$$

which indicates that

$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \ldots, \quad \alpha_{r-2} = \varepsilon_{r-2} - \varepsilon_{r-1}, \quad \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r, \quad \alpha_r = \varepsilon_{r-1} + \varepsilon_r,$

so that

DYNKIN DIAGRAM WITH multiple bond at α_{r-1} and α_r end,

and

$$R^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le r\}.$$

Then

$$\rho = \frac{1}{2}\sum \alpha \in R^+ = \frac{1}{2}\left(\sum_{1\le i<j\le r} (\varepsilon_i - \varepsilon_j) + (\varepsilon_i + \varepsilon_j)\right) = \frac{1}{2}\sum_{i=1}^{r}(2r-2i)\varepsilon_i$$

satisfies $\langle \rho, \alpha_i^\vee\rangle = \langle \rho, \varepsilon_i - \varepsilon_{i+1}\rangle = 1$, for $i = 1, 2, \ldots, r-1$ and

$$\langle \rho, \alpha_r^\vee\rangle = \langle \rho, \varepsilon_{r-1} + \varepsilon_r\rangle = 1.$$

Then the quantum dimension of $L(\varepsilon_1)$ is

$$
\begin{aligned}
&q^{\langle 2\rho, \varepsilon_1\rangle} + \cdots + q^{\langle 2\rho, \varepsilon_r\rangle} + q^{\langle 2\rho, -\varepsilon_r\rangle} + \cdots + q^{\langle 2\rho, -\varepsilon_1\rangle} \\
&\qquad = q^{2r-2} + q^{2r-4} + \cdots + q^{2r-2r} + q^{-(2r-2r)} + \cdots + q^{-(2r-2)} \\
&\qquad = q^{2r-2} + q^{2r-4} + \cdots + q^2 + 1 + 1 + q^{-2} + \cdots + q^{-(2r-2)} = [2r-1]+1.
\end{aligned}
$$

If $\lambda = \varepsilon_1$ then

$$\begin{aligned}
\prod_{\alpha\in R^+} \frac{[\langle\lambda+\rho,\alpha\rangle]}{[\langle\rho,\alpha\rangle]} &= \prod_{1\le i<j\le r} \frac{[\langle\lambda+\rho,\varepsilon_i-\varepsilon_j\rangle]}{[\langle\rho,\varepsilon_i-\varepsilon_j\rangle]}\frac{[\langle\lambda+\rho,\varepsilon_i+\varepsilon_j\rangle]}{[\langle\rho,\varepsilon_i+\varepsilon_j\rangle]} \\
&= \prod_{1\le i<j\le r} \frac{[\lambda_i-\lambda_j+j-i]}{[j-i]}\frac{[\lambda_i+\lambda_j+2r-i-j]}{[2r-i-j]} \\
&= \prod_{j=2} \frac{[2][3]\cdots[r]}{[1][2]\cdots[r-1]}\frac{[2r-2][2r-3]\cdots[r]}{[2r-3][2r-4]\cdots[r-1]} \\
&= \frac{[r]}{[1]}\frac{[2r-2]}{[r-1]} = \frac{(q^r-q^{-r})(q^{2r-2}-q^{-(2r-2)})}{(q-q^{-1})(q^{r-1}-q^{-(r-1)})} \\
&= \frac{(q^r-q^{-r})(q^{r-1}-q^{-(r-1)})}{q-q^{-1}} = \frac{q^{2r-1}+q-q^{-1}-q^{-(2r-1)}}{q-q^{-1}} = [2r-1]+1.
\end{aligned}$$

We'd also like to match up with [BB, Prop. 7.6] where

$$\dim_q(L(\lambda)) = \prod_{j=1}^{r} \frac{[r+\lambda_j-j+\frac{1}{2}]}{[r-j+\frac{1}{2}]}\cdot\prod_{1\le i<j\le r}\frac{[2r+\lambda_i-i+\lambda_j-j+1]}{[2r-i-j+1]}\frac{[\lambda_i-i-\lambda_j+j]}{[j-i]}, \qquad \text{for } \mathfrak{g}=\mathfrak{so}_{2r+1},$$

$$\dim_q(L(\lambda)) = (-1)^{|\lambda|}\prod_{j=1}^{r} \frac{[2r+2+2\lambda_j-2j]}{[2r+2-2j]}\cdot\prod_{1\le i<j\le r}\frac{[2r+2+\lambda_i-i+\lambda_j-j]}{[2r+2-i-j]}\frac{[\lambda_i-i-\lambda_j+j]}{[j-i]}, \qquad \text{for } \mathfrak{g}=\mathfrak{sp}_{2r},$$

$$\dim_q(L(\lambda)) = 2^{\delta_{\lambda_r 0}}\cdot\prod_{1\le i<j\le r}\frac{[2r+\lambda_i-i+\lambda_j-j]}{[2r-i-j]}\frac{[\lambda_i-i-\lambda_j+j]}{[j-i]}, \qquad \text{for } \mathfrak{g}=\mathfrak{so}_{2r},$$

As John Enyang has explained the relation between the central element generating functions $Z^{\pm}(u)$ and the q-dimension formulas is given by

$$(Z_{k-1})_{TT} = \sum_{S\diamond T}\frac{(E_{k-1})_{SS}}{u-(Y_{k-1})_{SS}}, \tag{2.1}$$

where $(a)_{ST}$ denotes the (S,T)-matrix entry of a in the seminormal representation (this formula is another way of stating Nazarov's residue formula). The conversion to the [BB, Prop. 7.6] formula should be determined by the identity (2.1).

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